Further Math with Ali Hashir

Vectors

Further Pure Mathematics 1 – Chapter 6 • CIE 9231

About this chapter

You already know how to write down the equation of a line in three dimensions, and how to use the scalar product to find an angle. This chapter adds one new tool and one new object, and almost everything else follows from them. The new tool is the *vector product* $\mathbf{a} \times \mathbf{b}$. Unlike the scalar product it returns a vector, and the point of it is that this vector is *perpendicular to both* of the originals. That single property is what makes it so useful: a normal to a plane through three points, the direction of the common perpendicular to two skew lines, the direction of the line where two planes meet — all of them are vector products. Its length carries information too, since $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

The new object is the *plane*. A plane can be described in three ways — by a point and two directions, by a point and a normal, or by a Cartesian equation — and a large part of the skill in this chapter is choosing the right one and converting between them fluently. After that the questions are all variations on three themes: *distances* (point to plane, point to line, between skew lines), *angles* (line to line, line to plane, plane to plane), and *intersections* (where a line meets a plane, where two planes meet). Almost every examination question is built from those pieces.

Key results: vector products, lines and planes

1. The vector product. For $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\,\mathbf{i} - (a_1b_3 - a_3b_1)\,\mathbf{j} + (a_1b_2 - a_2b_1)\,\mathbf{k}.$$

It is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$; $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$; and $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$, so the *order matters*. It is zero exactly when the two vectors are parallel.

2. Areas. $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$ is the area of the parallelogram on $\overrightarrow{AB}$ and $\overrightarrow{AC}$, so the area of triangle ABC is $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$.

3. A line through $\mathbf{a}$ with direction $\mathbf{b}$ is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$. Two lines are *parallel* if their directions are multiples of each other; otherwise they either *intersect* (one value of each parameter satisfies all three component equations) or are *skew* (no such values).

4. A plane may be written in any of three ways, all carrying the same information:

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$	a point and two directions in the plane
$\mathbf{r} \cdot \mathbf{n} = p$	a normal $\mathbf{n}$, with $p = \mathbf{a} \cdot \mathbf{n}$
$ax + by + cz = d$	the same thing written out, with $\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

To go from the first form to the others, take $\mathbf{n} = \mathbf{b} \times \mathbf{c}$. **The coefficients in the Cartesian equation are the components of the normal** — that one fact does most of the converting for you.

5. Building a plane. Through three points A, B, C: take $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$. Containing a line and a point P: take $\mathbf{n} = \mathbf{d} \times \overrightarrow{AP}$. Containing one line and parallel to another: $\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$. Containing a line and perpendicular to a plane: $\mathbf{n} = \mathbf{d} \times \mathbf{n}_1$.

Key results: distances, angles and intersections

6. Distances.

point (x_0, y_0, z_0) to the plane $ax + by + cz = d$	$\dfrac{\lvert ax_0 + by_0 + cz_0 - d\rvert}{\sqrt{a^2 + b^2 + c^2}}$
point P to the line through A with direction $\mathbf{d}$	$\dfrac{\left\lvert \overrightarrow{AP} \times \mathbf{d}\right\rvert}{\lvert\mathbf{d}\rvert}$
between the skew lines $\mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{a}_2 + \mu\mathbf{d}_2$	$\dfrac{\lvert(\mathbf{a}_2 - \mathbf{a}_1)\cdot\mathbf{n}\rvert}{\lvert\mathbf{n}\rvert}, \quad \mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2$

7. Angles. Between two *lines*, and between two *planes*, use the cosine; between a *line and a plane*, use the sine:

$$\cos\theta = \frac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1|\,|\mathbf{d}_2|}, \qquad \cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}, \qquad \sin\theta = \frac{|\mathbf{n} \cdot \mathbf{d}|}{|\mathbf{n}|\,|\mathbf{d}|}.$$

The sine appears in the third because $\mathbf{n}$ points *away* from the plane, not along it. The modulus signs are what make the answer the acute angle.

8. A line and a plane. Test $\mathbf{n}\cdot\mathbf{d}$:

$\mathbf{n}\cdot\mathbf{d} \neq 0$	the line *meets* the plane at exactly one point
$\mathbf{n}\cdot\mathbf{d} = 0$ and a point of the line satisfies the equation	the line *lies in* the plane
$\mathbf{n}\cdot\mathbf{d} = 0$ but the point does not	the line is *parallel* to the plane

To find the point of intersection, substitute the general point of the line into the equation of the plane and solve for the parameter.

9. Two planes that are not parallel meet in a line, whose direction is $\mathbf{n}_1 \times \mathbf{n}_2$. For a point on it, set one variable to zero and solve the two remaining equations.

10. Feet of perpendiculars. From P to a plane: write $N = P + t\mathbf{n}$, substitute into the equation of the plane and solve for t. From P to a line $\mathbf{a} + t\mathbf{d}$: require $\overrightarrow{PN}\cdot\mathbf{d} = 0$. In both cases the reflection of P is $2\,\overrightarrow{ON} - \overrightarrow{OP}$.

11. The common perpendicular to two skew lines meets them at P and Q. Write down the general points, form $\overrightarrow{PQ}$, and solve the two equations $\overrightarrow{PQ}\cdot\mathbf{d}_1 = 0$ and $\overrightarrow{PQ}\cdot\mathbf{d}_2 = 0$ simultaneously.

How to use this worksheet

- The twenty questions are in **increasing order of difficulty**: 1–7 cover vector products, lines and planes; 8–14 build up distances, intersections and angles; 15–20 are examination-length. Work through them in order.
- Marks are in square brackets. Allow about **$1\frac{1}{2}$ minutes per mark**; the full worksheet is roughly **4 hours** of work, so it is best split over two sittings.
- **Draw the vector product out in full**, with the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ row on top and the minus sign already written in front of the $\mathbf{j}$ term. Almost every lost mark in this chapter is a sign in a 2×2 determinant.
- **Check every plane on all the points you were given.** If the equation is supposed to pass through A, B and C, substitute all three. It takes seconds and catches an error before it contaminates the rest of the question.
- Before reaching for a formula, ask *which* object each vector belongs to: a direction lies *along* a line, a normal points *away* from a plane. That settles whether you need cos or sin, and whether to use $\mathbf{d}$ or $\mathbf{n}$.
- Simplify normals by dividing out common factors before going on, and leave answers **exact** — surds and fractions — unless the question asks for degrees or a number of decimal places.
- Where a question says *show that*, all the marks are in the working.

QUESTIONS 1–7 • Vector products, lines and planes

Question 1

The vectors $\mathbf{a}$ and $\mathbf{b}$ are given by $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

(a) Find $\mathbf{a} \times \mathbf{b}$. [2]

(b) Verify that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. [2]

(c) Find a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. [2]

Question 2

The points A, B and C have coordinates $(1, 0, 2)$, $(3, 2, 1)$ and $(2, -1, 4)$ respectively.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]

(b) Hence find the exact area of triangle ABC. [2]

(c) Hence find the exact perpendicular distance from C to the line AB. [3]

Question 3

The line l_1 passes through the points $P(2, -1, 3)$ and $Q(5, 1, -1)$.

(a) Find a vector equation for l_1. [2]

(b) Determine whether the point $R(8, 3, -5)$ lies on l_1. [2]

(c) The line l_2 has equation $\mathbf{r} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k} + s\,(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$. Find the acute angle between l_1 and l_2, giving your answer in degrees correct to 1 decimal place. [3]

Question 4

(a) The plane Π_1 passes through the point $A(1,2,1)$ and is perpendicular to $\mathbf{n} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Write down an equation for Π_1 in the form $\mathbf{r} \cdot \mathbf{n} = p$, and hence give its Cartesian equation. [3]

(b) Show that $\mathbf{i} + 2\mathbf{j}$ and $3\mathbf{j} + \mathbf{k}$ are both parallel to Π_1, and hence write down an equation for Π_1 in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [3]

(c) The plane Π_2 has equation

$$\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + s\,(\mathbf{i} + \mathbf{j} - \mathbf{k}) + t\,(2\mathbf{i} + \mathbf{k})\,.$$

Find a Cartesian equation for Π_2. [4]

Question 5

The points A, B and C have position vectors $\mathbf{i} + 2\mathbf{k}$, $2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $-\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ respectively.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz = d$. [4]

(b) Show that the point $D(4,5,-1)$ also lies in this plane. [1]

(c) Find the exact perpendicular distance from the point $E(3,2,5)$ to the plane. [3]

Question 6

(a) The line l_1 has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + t\,(2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ and the plane Π has equation $3x + 3y - z = 4$. Show that l_1 is parallel to Π but does not lie in it, and find the exact distance between them. [5]

(b) The line l_2 has equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + s\,(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. Find the coordinates of the point at which l_2 meets the plane $x - y + 2z = 15$. [3]

(c) Find the acute angle between l_2 and the plane $x - y + 2z = 15$. [3]

Question 7

The planes Π_1 and Π_2 have equations

$$\Pi_1 : \; 2x - y + 2z = 5, \qquad \Pi_2 : \; x + 2y + 2z = 9.$$

(a) Find the acute angle between Π_1 and Π_2, giving your answer in degrees correct to 1 decimal place. [3]

(b) Find the perpendicular distance from the point $P(4, 1, -2)$ to Π_1. [2]

(c) Find the coordinates of the foot of the perpendicular from P to Π_1. [4]

QUESTIONS 8–14 • Distances, intersections and angles

Question 8

The line l has equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + t\,(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, and the point P has coordinates $(3, 1, -2)$.

(a) Find the exact perpendicular distance from P to l. [4]

(b) Find the coordinates of the foot of the perpendicular from P to l. [4]

Question 9

The lines l_1 and l_2 have equations

$$l_1 : \; \mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda\,(2\mathbf{i} - \mathbf{j} + \mathbf{k})\,, \qquad l_2 : \; \mathbf{r} = 2\mathbf{i} + 3\mathbf{k} + \mu\,(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\,.$$

(a) Show that l_1 and l_2 are skew. [4]

(b) Find the acute angle between l_1 and l_2, giving your answer in degrees correct to 1 decimal place. [3]

(c) Find the exact shortest distance between l_1 and l_2. [4]

Question 10

The line l has equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + t\,(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ and the point P has coordinates $(3, 1, 4)$.

(a) Find the equation of the plane Π containing l and P, giving your answer in the form $ax + by + cz = d$. [5]

(b) Find the exact perpendicular distance from the point $Q(1, 4, -3)$ to Π. [3]

(c) Find the acute angle between Π and the plane $x + y + z = 1$, giving your answer in degrees correct to 1 decimal place. [3]

Question 11

The planes Π_1 and Π_2 have equations

$$\Pi_1 : \ x + y + 2z = 4, \qquad \Pi_2 : \ 2x - y + z = 5.$$

(a) Find the acute angle between Π_1 and Π_2. [3]

(b) Find a vector equation for the line of intersection l of Π_1 and Π_2. [5]

(c) Hence find the equation of the plane that contains l and is perpendicular to Π_1, giving your answer in the form $ax + by + cz = d$. [4]

Question 12

The plane Π has equation $x + 2y + 2z = 5$ and the point P has coordinates $(4, 3, 2)$.

(a) Find the perpendicular distance from P to Π. [2]

(b) Find the coordinates of N, the foot of the perpendicular from P to Π. [4]

(c) Find the coordinates of P', the reflection of P in Π. [2]

(d) Show that the line $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t\,(2\mathbf{i} - \mathbf{j})$ lies in Π. [3]

Question 13

The lines l_1 and l_2 have equations

$$l_1: \ \mathbf{r} = \mathbf{i} - 3\mathbf{j} + a\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k}), \qquad l_2: \ \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mu(2\mathbf{i} - \mathbf{j} + 2\mathbf{k}),$$

where a is a constant. It is given that l_1 and l_2 intersect.

(a) Find the value of a. [4]

(b) Find the position vector of the point of intersection. [2]

(c) Find the acute angle between l_1 and l_2. [3]

(d) Find the equation of the plane containing l_1 and l_2, giving your answer in the form $ax + by + cz = d$. [4]

Question 14

The lines l_1 and l_2 have equations

$$l_1: \ \mathbf{r} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k}), \qquad l_2: \ \mathbf{r} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

(a) Show that l_1 and l_2 are skew. [4]

(b) The plane Π contains l_1 and is parallel to l_2. Find the equation of Π in the form $ax + by + cz = d$. [4]

(c) Hence find the exact shortest distance between l_1 and l_2. [3]

(d) Find the acute angle between Π and the plane $2x + y - 2z = 7$. [3]

QUESTIONS 15–20 • Examination-standard problems

Question 15

The points A, B, C and D have position vectors

$$\mathbf{i}+\mathbf{j}+2\mathbf{k}, \qquad 3\mathbf{i}+\mathbf{k}, \qquad 2\mathbf{j}+4\mathbf{k}, \qquad 2\mathbf{i}+3\mathbf{j},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the exact perpendicular distance from D to the plane ABC. [2]

(c) Find the acute angle between the planes ABC and ABD. [4]

(d) Find the exact shortest distance between the lines AB and CD. [5]

Question 16

The line l_1 passes through the point $A(1,2,0)$ and is parallel to $\mathbf{i}+\mathbf{j}+\mathbf{k}$. The line l_2 passes through the point $B(3,0,t)$ and is parallel to $\mathbf{i}-\mathbf{j}$, where t is a constant.

(a) Show that the shortest distance between l_1 and l_2 is $\frac{|2t|}{\sqrt{6}}$, and deduce that the lines intersect when $t=0$. [5]

(b) Given instead that the shortest distance between l_1 and l_2 is $\sqrt{6}$, find the two possible values of t. [3]

(c) Taking $t=3$, find the equation of the plane Π that contains l_1 and is parallel to l_2, giving your answer in the form $ax+by+cz=d$. [3]

(d) Find the acute angle between l_2 and the plane $2x-y+2z=1$. [3]

Question 17

The plane Π has equation

$$\mathbf{r}=\mathbf{i}+\mathbf{j}+\lambda\left(2\mathbf{i}-\mathbf{j}+\mathbf{k}\right)+\mu\left(\mathbf{i}+2\mathbf{j}-\mathbf{k}\right).$$

(a) Find a Cartesian equation for Π in the form $ax+by+cz=d$. [4]

(b) Find the exact perpendicular distance from the point $P(2,1,3)$ to Π. [3]

(c) Find the position vector of the foot of the perpendicular from P to Π. [4]

Question 18

The lines l_1 and l_2 have equations

$$l_1: \ \mathbf{r} = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} + \lambda\,(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}), \qquad l_2: \ \mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + \mu\,(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}).$$

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both lines.

(a) Find the exact shortest distance between l_1 and l_2. [4]

(b) Find the position vectors of P and Q. [7]

(c) Find a vector equation for the line PQ. [2]

(d) Verify that $\left|\overrightarrow{PQ}\right|$ agrees with your answer to part (a). [2]

Question 19

The plane Π_1 has equation

$$\mathbf{r} = 2\mathbf{i} + \mathbf{k} + s\,(\mathbf{i} + 2\mathbf{k}) + u\,(\mathbf{j} - \mathbf{k}),$$

and the plane Π_2 has equation $x + y + 2z = 6$.

(a) Find a Cartesian equation for Π_1 in the form $ax + by + cz = d$. [4]

(b) Find a vector equation for the line of intersection of Π_1 and Π_2. [4]

(c) Find the exact perpendicular distance from the point $R(1, -2, 3)$ to Π_1. [3]

(d) Show that the line $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + v\,(\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$ is parallel to Π_1, and find the exact distance between them. [4]

Question 20

The points A, B, C and D have coordinates $(1, 2, 1)$, $(3, 1, 2)$, $(2, 4, 3)$ and $(0, 1, 4)$ respectively.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$, and hence the exact area of triangle ABC. [4]

(b) Find the equation of the plane ABC in the form $ax + by + cz = d$. [2]

(c) Find the exact perpendicular distance from D to the plane ABC. [3]

(d) Hence find the volume of the tetrahedron $ABCD$. *(You may use the fact that the volume of a tetrahedron is $\frac{1}{3}\times$ area of base $\times$ perpendicular height.)* [2]

(e) Find the acute angle between the planes ABC and ABD. [4]