Further Math with Ali Hashir

Summation of Series

Further Pure Mathematics 1 – Chapter 3 • CIE 9231

About this chapter

Adding up a list of numbers is easy; adding up a list whose length is the unknown n needs a different idea. This chapter supplies two of them, and they pull in opposite directions.

The first is to *build* a sum out of ones you already know. Three results — $\sum r$, $\sum r^2$, $\sum r^3$ — are given to you in MF19, and because summation is linear, any polynomial in r can be summed by splitting it into those pieces. The second is the *method of differences*, and it works the other way round: if each term can be written as the difference of consecutive values of some function, then almost everything cancels when the terms are added, and only a handful survives at each end. Partial fractions are usually how you force a term into that shape. What is left over at the far end also answers the last question of the chapter — whether the sum settles down to a limit as n grows, and how large n must be before it is close enough.

Key results you will need

1. The standard results (quotable from MF19)

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

2. Summation is linear: $\sum\big(a\,\mathrm{f}(r) + b\,\mathrm{g}(r)\big) = a\sum \mathrm{f}(r) + b\sum \mathrm{g}(r)$ over the same range, and a constant summed n times gives $\sum_{r=1}^{n} c = cn$. So any polynomial in r splits into the three results above.

3. Changing the range: to start somewhere other than $r = 1$, subtract the part you do not want, $\sum_{r=m}^{n} \mathrm{f}(r) = \sum_{r=1}^{n} \mathrm{f}(r) - \sum_{r=1}^{m-1} \mathrm{f}(r)$.

4. The method of differences. If $u_r = \mathrm{f}(r) - \mathrm{f}(r+1)$, then adding from $r = 1$ to n leaves only the ends:

$$\sum_{r=1}^{n} u_r = \mathrm{f}(1) - \mathrm{f}(n+1).$$

Write out the first two and the last two terms in full: that is how you see which survive. If the gap between the factors is bigger than one, more than one term survives at each end.

5. Sum to infinity. The series converges exactly when the surviving end terms tend to zero, and then $S = \lim_{n\to\infty} S_n$. Those same leftover terms *are* $S - S_n$, which is what a "least value of n" question is asking you to make small.

How to use this worksheet

- The twenty questions are in **increasing order of difficulty**: 1–7 rehearse the standard results, 8–14 build the method of differences, 15–20 are examination-length. Work through them in order.
- Marks are in square brackets. Allow about **$1\frac{1}{2}$ minutes per mark**; the full worksheet is roughly **3 hours** of work.
- The three standard results may be quoted from MF19 without proof — except in the two questions that ask you to *derive* one of them, where quoting it earns nothing.
- In a method-of-differences answer, **show the cancellation**. Write out enough terms at each end that it is clear what is left; a bare final answer earns few marks.
- Leave every answer **exact**, and factorise where the question asks for a simplified form.
- Where a question says *show that*, all the marks are in the working.

QUESTIONS 1–7 • Standard results

Question 1

Use the standard results for $\sum r$, $\sum r^2$ and $\sum r^3$ to find, in terms of n, an expression for each of the following.

(a) $\displaystyle\sum_{r=1}^{n}(4r+3)$ [2]

(b) $\displaystyle\sum_{r=1}^{n}(6r^2-2)$ [2]

(c) $\displaystyle\sum_{r=1}^{n}(4r^3+1)$ [2]

Question 2

Show that

$$\sum_{r=1}^{n}(2r-7)=n(n-6).$$

[3]

Question 3

Show that

$$\sum_{r=1}^{n}r(r+4)=\tfrac{1}{6}n(n+1)(2n+13).$$

[4]

Question 4

Find $\displaystyle\sum_{r=1}^{n}(3r-1)(r+2)$, giving your answer in its simplest form.

[4]

Question 5

Find, in terms of n, an expression for

$$\sum_{r=n+1}^{3n} r^2.$$

[4]

Question 6

Determine the value of

$$1 - 3 + 5 - 7 + \cdots + (4n-3) - (4n-1).$$

[3]

Question 7

Without using a calculator, find the sum of the squares of the first 50 odd positive integers.

[4]

QUESTIONS 8–14 • The method of differences

Question 8

(a) Show that

$$\frac{1}{(4r-1)(4r+3)} = \frac{1}{4}\left(\frac{1}{4r-1} - \frac{1}{4r+3}\right).$$

[2]

(b) Hence use the method of differences to find $\displaystyle\sum_{r=1}^{n} \frac{1}{(4r-1)(4r+3)}$ in terms of n. [3]

(c) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(4r-1)(4r+3)}$. [1]

Question 9

(a) Express $\dfrac{1}{r(r+3)}$ in partial fractions. [2]

(b) Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+3)}$ in terms of n. [4]

(c) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r(r+3)}$. [1]

Question 10

Let $\mathrm{f}(r) = r(r+1)(r+2)(r+3)$.

(a) Show that $\mathrm{f}(r) - \mathrm{f}(r-1) = 4r(r+1)(r+2)$. [2]

(b) Hence show that $\displaystyle\sum_{r=1}^{n} r(r+1)(r+2) = \tfrac{1}{4}n(n+1)(n+2)(n+3)$. [3]

Question 11

Using the result of Question 8, or otherwise, find

$$\sum_{r=n+1}^{2n} \frac{1}{(4r-1)(4r+3)}$$

in terms of n, simplifying your answer.

[4]

Question 12

(a) Show that $(r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$. [2]

(b) Hence, using the standard results for $\sum r$ and $\sum r^2$, prove that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

[4]

Question 13

Let $S_n = \sum_{r=1}^{n} (-1)^r r^3$.

(a) Show that $S_{2n} = n^2(4n+3)$. [4]

(b) Hence find S_{2n+1}, factorising your answer. [3]

Question 14

Let $S_N = \sum_{r=1}^{N} \frac{1}{(5r-4)(5r+1)}$, and let $S = \lim_{N\to\infty} S_N$.

(a) Show that $S_N = \frac{N}{5N+1}$, and write down the value of S. [3]

(b) Find the least value of N for which $S - S_N < \frac{1}{1000}$. [3]

QUESTIONS 15–20 • Examination-standard problems

Question 15

(a) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}(r+1)(r+4) = \tfrac{1}{3}n(n+4)(n+5).$$

[3]

(b) Express $\dfrac{1}{(r+1)(r+4)}$ in partial fractions and hence use the method of differences to find $\displaystyle\sum_{r=1}^{n}\frac{1}{(r+1)(r+4)}$ in terms of n. [4]

(c) Deduce the value of $\displaystyle\sum_{r=1}^{\infty}\frac{1}{(r+1)(r+4)}$. [1]

Question 16

(a) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}(2r+1)(2r+5) = \tfrac{1}{3}n(2n+5)(2n+7).$$

[3]

(b) Use the method of differences to find $\displaystyle\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+5)}$ in terms of n. [4]

(c) Deduce the value of $\displaystyle\sum_{r=1}^{\infty}\frac{1}{(2r+1)(2r+5)}$. [1]

Question 17

(a) Use the method of differences to show that

$$\sum_{r=1}^{n}\ln\left(\frac{r+2}{r}\right) = \ln\left(\frac{(n+1)(n+2)}{2}\right).$$

[4]

(b) Explain why the infinite series $\ln\left(\frac{3}{1}\right)+\ln\left(\frac{4}{2}\right)+\ln\left(\frac{5}{3}\right)+\cdots$ does not converge. [2]

Question 18

The sum $S_n = \sum_{r=1}^{n} u_r$ is given by $S_n = \frac{1}{3}n(n+1)(n+5)$.

(a) Find u_r in terms of r, simplifying your answer. [3]

(b) Find $\sum_{r=n+1}^{2n} u_r$ in terms of n. [4]

Question 19

Let

$$S_n = \sum_{r=1}^{n}(6r-5)(6r+1) \qquad \text{and} \qquad T_n = \sum_{r=1}^{n} \frac{1}{(6r-5)(6r+1)}.$$

(a) Use standard results from the list of formulae (MF19) to show that $S_n = n\left(12n^2+6n-11\right)$. [3]

(b) Use the method of differences to show that $T_n = \dfrac{n}{6n+1}$. [4]

(c) Find $\lim_{n\to\infty} \dfrac{S_n T_n}{n^3}$. [2]

Question 20

Let $S_n = \sum_{r=1}^{n} \dfrac{1}{(2r-1)(2r+1)(2r+3)}$.

(a) Show that

$$\frac{1}{(2r-1)(2r+1)(2r+3)} = \frac{1}{4}\left(\frac{1}{(2r-1)(2r+1)} - \frac{1}{(2r+1)(2r+3)}\right).$$

[3]

(b) Hence show that $S_n = \dfrac{1}{12} - \dfrac{1}{4(2n+1)(2n+3)}$. [4]

(c) Write down the value of $S = \lim_{n\to\infty} S_n$. [1]

(d) Find the least value of n for which $S - S_n < 10^{-4}$. [4]

Answers and fully worked solutions are available in the companion booklets.

Instagram: **@vectorspacepk**
Website: **www.vectorspace.pk**