Further Math with Ali Hashir

Roots of Polynomial Equations

Further Pure Mathematics 1 – Chapter 1 • CIE 9231

About this chapter

Every polynomial equation carries a hidden record of its own roots. You do not need to solve $x^3 - 7x^2 + 2x - 3 = 0$ to know that its roots add up to 7 and multiply to give 3: the coefficients already say so. That one observation is the whole of this chapter, and it lets you evaluate expressions such as $\alpha^2 + \beta^2 + \gamma^2$ or $\sum 1/\alpha$ for equations you could never solve.

Two techniques do all the work. *Symmetric functions*: an expression in the roots that is unchanged when the roots are permuted among themselves can always be rebuilt from the elementary sums — $\sum \alpha$, then $\sum \alpha\beta$, and so on down to the product of all the roots — and each of those is a ratio of coefficients. *Substitution*: to find the equation satisfied by new roots $y = g(\alpha)$, rearrange $y = g(x)$ to make x the subject and put the result into the original equation. Each original root then contributes a root $g(\alpha)$ to the new equation, whose coefficients hand you the symmetric functions of the transformed roots immediately.

Key results you will need

1. Relations between roots and coefficients

$$ax^2 + bx + c = 0, \text{ roots } \alpha, \beta \qquad \sum \alpha = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}$$

$$ax^3 + bx^2 + cx + d = 0, \text{ roots } \alpha, \beta, \gamma \qquad \sum \alpha = -\frac{b}{a}, \quad \sum \alpha\beta = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}$$

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \qquad \sum \alpha = -\frac{b}{a}, \quad \sum \alpha\beta = \frac{c}{a},$$

$$\text{roots } \alpha, \beta, \gamma, \delta \qquad \sum \alpha\beta\gamma = -\frac{d}{a}, \quad \alpha\beta\gamma\delta = \frac{e}{a}$$

2. The identity that does most of the work (valid for every degree)

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta.$$

3. Power sums. Write S_n for the sum of the nth powers of the roots, so that S_0 is the degree. Adding the equation over all the roots gives a recurrence; for a cubic $ax^3 + bx^2 + cx + d = 0$ it is $aS_{n+3} + bS_{n+2} + cS_{n+1} + dS_n = 0$, and this holds for negative n too, which is how $S_{-1}, S_{-2}, \ldots$ are found.

4. Substitution. For roots $y = g(\alpha)$, make x the subject and substitute into the original equation; for $y = \alpha^2$ isolate the odd powers on one side before squaring.

How to use this worksheet

- Questions are in **increasing order of difficulty**: 1–10 rehearse the standard results, 11–20 build the substitution and S_n techniques, 21–30 are examination-length. Work through them in order.
- Marks are in square brackets. Allow about **$1\frac{1}{2}$ minutes per mark**, so the full worksheet is roughly **3 hours** of work.
- Write down the elementary symmetric sums of the roots *first* — all of them, for the degree you are given. Almost every part is then a short piece of algebra.
- Leave every answer **exact**; do not find the roots numerically on a calculator.
- Where a question says *show that*, all the marks are in the working, so set it out in full. You may quote $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ from MF19.

QUESTIONS 1–10 • Foundations: reading the coefficients

Question 1

For each of the following quadratic equations the roots are α and β. Write down the value of $\alpha + \beta$ and the value of $\alpha\beta$.

(a) $x^2 - 7x + 5 = 0$ [2]

(b) $3x^2 + 8x - 2 = 0$ [2]

(c) $5x^2 - 4 = 0$ [2]

Question 2

For each of the following cubic equations the roots are α, β and γ. Write down the values of $\sum \alpha$, $\sum \alpha\beta$ and $\alpha\beta\gamma$.

(a) $x^3 + 4x^2 - 2x + 9 = 0$ [3]

(b) $2x^3 - 5x^2 + x + 3 = 0$ [3]

(c) $4x^3 + 7x - 1 = 0$ [3]

Question 3

The quartic equation

$$3x^4 - 2x^3 + 5x^2 + x - 8 = 0$$

has roots α, β, γ and δ. Write down the values of

$$\sum \alpha, \qquad \sum \alpha\beta, \qquad \sum \alpha\beta\gamma, \qquad \alpha\beta\gamma\delta.$$

[4]

Question 4

Find, with integer coefficients,

(a) the quadratic equation whose roots are $\frac{3}{2}$ and -4, [2]

(b) the cubic equation whose roots are -1, 2 and 5, [3]

(c) the quartic equation whose roots are 1, -3, $1+2\mathrm{i}$ and $1-2\mathrm{i}$. [4]

Question 5

Find the value of the sum of the squares of the roots of each of the following equations.

(a) $x^3 - 3x^2 + 8x + 5 = 0$ [2]

(b) $2x^4 - 4x^3 + x^2 + 6x - 1 = 0$ [3]

Question 6

The quartic equation

$$2x^4 + x^3 - 6x^2 + 4x - 5 = 0$$

has roots α, β, γ and δ. Find the value of

(a) $\sum \frac{1}{\alpha}$, [3]

(b) $\sum \frac{1}{\alpha\beta}$. [2]

Question 7

The quadratic equation $2x^2 - 6x + 5 = 0$ has roots α and β. Find, with integer coefficients, the quadratic equation whose roots are

(a) $\alpha - 3$ and $\beta - 3$, [3]

(b) 5α and 5β, [2]

(c) α^2 and β^2. [3]

Question 8

The cubic equation

$$x^3 + px^2 + qx - 12 = 0,$$

where p and q are constants, has roots α, β and γ. It is given that

$$\alpha + \beta + \gamma = 5 \qquad \text{and} \qquad \alpha^2 + \beta^2 + \gamma^2 = 13.$$

(a) Write down the value of p. [1]

(b) Find the value of q. [3]

(c) Hence find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. [2]

Question 9

The cubic equation

$$3x^3 - 2x^2 + 5x - 4 = 0$$

has roots α, β and γ.

(a) Find, with integer coefficients, the cubic equation whose roots are $\dfrac{1}{\alpha}$, $\dfrac{1}{\beta}$ and $\dfrac{1}{\gamma}$. [3]

(b) Hence find the value of $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2}$. [2]

Question 10

The cubic equation

$$x^3 + 5x^2 - 2x + 7 = 0$$

has roots α, β and γ.

(a) Use the substitution $y = x + 2$ to find the cubic equation whose roots are $\alpha + 2$, $\beta + 2$ and $\gamma + 2$. [4]

(b) Hence write down the value of $(\alpha + 2)(\beta + 2)(\gamma + 2)$. [1]

QUESTIONS 11–20 • Substitutions and power sums

Question 11

The cubic equation

$$x^3 - 4x^2 + x - 6 = 0$$

has roots α, β and γ.

(a) Find a cubic equation whose roots are α^2, β^2 and γ^2, giving your answer in the form $y^3 + py^2 + qy + r = 0$. [4]

(b) Write down the value of $\alpha^2+\beta^2+\gamma^2$ and verify it directly from the original equation. [2]

(c) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]

Question 12

The cubic equation

$$2x^3 - x^2 + 4x - 3 = 0$$

has roots α, β and γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) Write down the value of S_1 and find the value of S_2. [3]

(b) Show that $2S_{n+3} - S_{n+2} + 4S_{n+1} - 3S_n = 0$. [2]

(c) Hence find the values of S_3 and S_4. [4]

Question 13

The cubic equation

$$x^3 + 2x^2 - 5x + 4 = 0$$

has roots α, β and γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) Find the value of S_{-1}. [2]

(b) Find the value of S_{-2}. [3]

(c) Find the value of S_{-3}. [3]

Question 14

The roots of the cubic equation

$$x^3 - 12x^2 + 39x - 28 = 0$$

are in arithmetic progression. Find the three roots.

[5]

Question 15

The roots of the cubic equation

$$8x^3 - 42x^2 + kx - 27 = 0,$$

where k is a constant, are in geometric progression.

(a) Find the three roots. [6]

(b) Deduce the value of k. [2]

Question 16

The cubic equation

$$3x^3 - 2x^2 + x + 4 = 0$$

has roots α, β and γ. Find the value of

(a) $(1+\alpha)(1+\beta)(1+\gamma)$, [3]

(b) $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$. [4]

Question 17

The cubic equation

$$x^3 - 2x + 5 = 0$$

has roots α, β and γ.

(a) Find a cubic equation whose roots are α^3, β^3 and γ^3, and state the value of $\alpha^3 + \beta^3 + \gamma^3$. [4]

(b) Find the value of $\alpha^6 + \beta^6 + \gamma^6$. [2]

(c) Find the value of $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3} + \dfrac{1}{\gamma^3}$. [2]

Question 18

The quartic equation

$$x^4 + 2x^3 - x + 3 = 0$$

has roots α, β, γ and δ.

(a) Find a quartic equation whose roots are α^2, β^2, γ^2 and δ^2, and state the value of $\sum \alpha^2$. [5]

(b) Find the value of $\sum \alpha^4$. [2]

(c) Find the value of $\sum \dfrac{1}{\alpha^2}$. [2]

Question 19

The numbers α, β and γ satisfy

$$\alpha + \beta + \gamma = 4, \qquad \alpha^2 + \beta^2 + \gamma^2 = 6, \qquad \alpha^3 + \beta^3 + \gamma^3 = 10.$$

(a) Find the cubic equation with roots α, β and γ, giving your answer in the form $x^3 + px^2 + qx + r = 0$ where p, q and r are integers. [6]

(b) Hence write down the three numbers. [2]

Question 20

By forming a suitable cubic equation, solve the simultaneous equations

$$\alpha+\beta+\gamma=6, \qquad \alpha^2+\beta^2+\gamma^2=14, \qquad \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{11}{6}.$$

[8]

QUESTIONS 21–30 • Examination-standard problems

Question 21

The cubic equation

$$2x^3+3x^2-x+4=0$$

has roots α, β and γ.

(a) Find, with integer coefficients, the cubic equation whose roots are

$$\frac{\alpha-1}{\alpha}, \qquad \frac{\beta-1}{\beta}, \qquad \frac{\gamma-1}{\gamma}.$$

[5]

(b) Verify your answer by evaluating $\frac{\alpha-1}{\alpha}+\frac{\beta-1}{\beta}+\frac{\gamma-1}{\gamma}$ directly from the original equation. [3]

Question 22

The cubic equation

$$x^3 - 4x^2 + x + 7 = 0$$

has roots α, β and γ.

(a) Use the substitution $y = \dfrac{1}{x+1}$ to show that

$$y^3 + 12y^2 - 7y + 1 = 0$$

has roots $\dfrac{1}{\alpha+1}$, $\dfrac{1}{\beta+1}$ and $\dfrac{1}{\gamma+1}$. [4]

(b) Hence find the value of $\sum \dfrac{1}{(\alpha+1)^2}$. [3]

(c) Deduce the value of $(\alpha+1)(\beta+1)(\gamma+1)$, and hence find the value of $\sum(\beta+1)(\gamma+1)$. [3]

Question 23

The quartic equation

$$x^4 + x^3 - 2x + 3 = 0$$

has roots α, β, γ and δ.

(a) Show that a quartic equation with roots $2\alpha - 1$, $2\beta - 1$, $2\gamma - 1$ and $2\delta - 1$ is

$$y^4 + 6y^3 + 12y^2 - 6y + 35 = 0.$$

[4]

The sum $(2\alpha - 1)^n + (2\beta - 1)^n + (2\gamma - 1)^n + (2\delta - 1)^n$ is denoted by S_n.

(b) Find the value of S_2. [2]

(c) Given that $S_3 = 18$, find the value of S_4. [3]

Question 24

The cubic equation

$$x^3 + px + q = 0,$$

where p and q are non-zero real constants, has a repeated root.

(a) Show that $4p^3 + 27q^2 = 0$. [6]

(b) Given also that $p = -12$ and $q > 0$, find the value of q and solve the equation completely. [4]

Question 25

The quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

has roots α, $-\alpha$, β and $-\beta$, where $\alpha\beta \neq 0$.

(a) Show that $a = 0$ and $c = 0$, and express b and d in terms of α and β. [5]

(b) Given further that $\alpha^2 + \beta^2 = 10$ and $\alpha^2\beta^2 = 9$, write down the equation and solve it completely. [4]

Question 26

The cubic equation

$$x^3 + 2x^2 + 5x + k = 0,$$

where k is a real constant, has roots α, β and γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Explain why the equation cannot have three real roots, whatever the value of k. [2]

(c) It is now given that two of the roots are purely imaginary. Find the value of k and state the three roots. [5]

Question 27

The cubic equation

$$x^3 - 2x^2 + 4x - 1 = 0$$

has roots α, β and γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Using standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that

$$\sum_{r=1}^{n} \left[(\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2 \right] = \tfrac{1}{2} n \left(2n^2 + an + b \right),$$

where a and b are integers to be determined. [6]

Question 28

The cubic equation

$$x^3 + bx^2 + cx + d = 0$$

has roots α, β and γ, where $\gamma = \alpha + \beta$.

(a) Show that $b^3 - 4bc + 8d = 0$. [5]

(b) Given that $b = -10$ and $d = -30$, find the value of c and hence solve the equation completely. [5]

Question 29

The quartic equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

where $p \neq 0$, has roots α, β, γ and δ such that $\alpha\beta = \gamma\delta$.

(a) Show that $r^2 = p^2 s$. [6]

(b) The quartic equation

$$x^4 - 4x^3 + 7x^2 - 8x + 4 = 0$$

has roots satisfying $\alpha\beta = \gamma\delta$. Find the value of $\alpha\beta$ and hence solve the equation completely. [6]

Question 30

The cubic equation

$$x^3 + 3x - 1 = 0$$

has roots α, β and γ.

(a) Use the substitution $y = \dfrac{1}{1+x}$ to show that the equation with roots $\dfrac{1}{1+\alpha}$, $\dfrac{1}{1+\beta}$ and $\dfrac{1}{1+\gamma}$ is

$$5y^3 - 6y^2 + 3y - 1 = 0.$$

[4]

(b) Find the value of $\sum \dfrac{1}{(1+\alpha)^2}$ and the value of $\sum \dfrac{1}{(1+\alpha)^3}$. [5]

(c) Show that $(1+\alpha)(1+\beta)(1+\gamma) = 5$, and hence find the exact value of

$$\frac{1+\alpha}{(1+\beta)(1+\gamma)} + \frac{1+\beta}{(1+\gamma)(1+\alpha)} + \frac{1+\gamma}{(1+\alpha)(1+\beta)}.$$

[5]