Further Math with Ali Hashir

Proof by Induction

Further Pure Mathematics 1 – Chapter 7 • CIE 9231

About this chapter

Every other chapter in this course asks you to *find* something. This one asks you to *prove* something — and it gives you a single tool for doing it, which works on an infinity of statements at once.

The idea is a chain of dominoes. If you can knock the first one over, and if you can show that *whenever* one falls the next one falls too, then every domino falls — and you never have to check them one at a time. So an induction proof has exactly two pieces of real work: the **base case**, showing the statement is true for the first value of n; and the **inductive step**, showing that *if* it is true for some $n = k$, *then* it is true for $n = k + 1$. The step is where the mathematics lives, and it always turns on the same move: take the statement for $k + 1$, find the statement for k hiding inside it, and replace that part using the assumption.

What varies from question to question is only *what* you are proving — a summation formula, a power of a matrix, an inequality, a divisibility claim, an nth derivative. This worksheet is grouped by those types — five questions on each — because each has its own characteristic manoeuvre in the inductive step, and they are much easier to learn a few at a time than all mixed together. A final mixed section then draws on all of them. Sometimes you are not even given the result: you try a few values, spot the pattern, conjecture a formula, and then prove it. That last skill is explicitly in the syllabus, and it is the one that feels most like doing mathematics rather than reproducing it.

Key results: the method

1. The structure of every proof. To prove a statement $P(n)$ for all $n \geq n_0$:

Base case	show $P(n_0)$ is true, usually by evaluating both sides
Assumption	assume $P(k)$ is true for some integer $k \geq n_0$
Inductive step	using that assumption, prove $P(k+1)$
Conclusion	state that $P(n)$ is therefore true for all $n \geq n_0$

The conclusion is worth a mark on its own in nearly every mark scheme. Write it out in words every time: *"since it is true for $n = n_0$, and true for $n = k+1$ whenever it is true for $n = k$, it is true for all $n \geq n_0$ by mathematical induction."*

2. Summations. Split off the last term:

$$\sum_{r=1}^{k+1} \mathrm{f}(r) = \underbrace{\sum_{r=1}^{k} \mathrm{f}(r)}_{\text{use the assumption}} + \mathrm{f}(k+1).$$

Then aim squarely for the target: write down what the given formula becomes at $n = k+1$ before you start simplifying, so you know what you are steering towards. Take out common factors early rather than expanding everything.

3. Matrices. Use $\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A}$ (or $\mathbf{A}\mathbf{A}^k$), replace $\mathbf{A}^k$ by the assumed form, and multiply out. The four entries are four separate small pieces of algebra — do them one at a time.

4. Comparisons. From $\mathrm{f}(k) > \mathrm{g}(k)$, build up to $\mathrm{f}(k+1)$ in stages: apply the assumption first, then show the resulting expression is at least $\mathrm{g}(k+1)$. Say clearly *which* inequality you are using at each step, and watch the direction: multiplying by a negative quantity reverses it.

5. Divisibility. To show $f(n)$ is divisible by d, write the assumption as $f(k) = d\,m$ for some integer m, then show

$$f(k+1) = d \times (\text{an integer}),$$

usually by producing $f(k+1) - f(k)$, or $f(k+1) - a\,f(k)$ for a suitable constant a, and showing that this is a multiple of d. The words "for some integer m" matter: without them nothing has been proved.

6. Derivatives. Differentiate the assumed nth derivative once more and show the answer is the given formula with k replaced by $k+1$. Product rule and chain rule do all the work; the difficulty is bookkeeping, not calculus.

7. Conjecture first. Where no formula is given, compute the first three or four cases, look for the pattern, state your conjecture clearly — and only then prove it. A pattern seen in four cases is not a proof, which is precisely why the induction is needed.

8. Two base cases. If a recurrence uses the *two* previous terms, the inductive step needs both $P(k)$ and $P(k+1)$ to deduce $P(k+2)$ — so two base cases must be checked, not one.

How to use this worksheet

- The thirty-two questions are grouped into **six sections by type of proof**: five each on summations, matrices, comparisons, divisibility and derivatives, then seven mixed questions. Within each section they run from easiest to hardest, and working straight through a section is the fastest way to learn its characteristic move.
- Marks are in square brackets. Induction is quicker per mark than most topics once the structure is fluent — allow about $\mathbf{1\frac{1}{4}}$ **minutes per mark**, making the full worksheet roughly $\mathbf{4\frac{1}{2}}$ **hours** of work. It is best done a section at a time rather than in one sitting.
- **Write the proof out in full every time**, including the base case and the concluding sentence. Induction is marked on its structure as much as its algebra, and the marks for those two are the easiest in the whole paper to collect — and the easiest to forget.
- **Say what you are assuming.** "Assume the result is true for $n = k$" is a line in the proof, not something to leave implied. So is the phrase "for some integer m" in a divisibility proof.
- Before starting the inductive step, **write down the target** — the given statement with n replaced by $k+1$. Knowing where you are going turns most of these proofs into a short piece of routine algebra.
- Check the **starting value**. Several questions here are false for small n and begin at $n = 2$, $n = 3$ or $n = 4$; the base case must be the first value claimed, not automatically $n = 1$.
- Leave every answer **exact**, and factorise where a factorised form is what the question states.

SECTION A • QUESTIONS 1–5 • Summations

Question 1

Prove by mathematical induction that

$$\sum_{r=1}^{n} (4r - 1) = n(2n + 1)$$

for all positive integers n.

[4]

Question 2

Prove by mathematical induction that

$$\sum_{r=1}^{n} (2r - 1)^2 = \tfrac{1}{3} n(2n - 1)(2n + 1)$$

for all positive integers n.

[5]

Question 3

(a) Prove by mathematical induction that

$$\sum_{r=1}^{n} \frac{1}{(3r - 2)(3r + 1)} = \frac{n}{3n + 1}$$

for all positive integers n. [6]

(b) State the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(3r - 2)(3r + 1)}$. [1]

Question 4

Prove by mathematical induction that

$$\sum_{r=1}^{n} r\,2^r = (n-1)\,2^{n+1} + 2$$

for all positive integers n.

[6]

Question 5

Prove by mathematical induction that

$$\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$$

for all positive integers n.

[6]

SECTION B • QUESTIONS 6–10 • Matrices

Question 6

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & 0 \\ -2n & 1 \end{pmatrix}.$$

[5]

Question 7

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

(a) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{pmatrix}.$$

[6]

(b) Find, in terms of n, the inverse of $\mathbf{A}^n$. [2]

Question 8

The matrix $\mathbf{B}$ is given by $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

(a) Find $\mathbf{B}^2$ and $\mathbf{B}^3$. [2]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{B}^n = \begin{pmatrix} 2^n & n\,2^{n-1} \\ 0 & 2^n \end{pmatrix}.$$

[6]

Question 9

The matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$.

Prove by mathematical induction that, for all positive integers n,

$$\mathbf{M}^n = \begin{pmatrix} 2^n + n\,2^{n-1} & -n\,2^{n-1} \\ n\,2^{n-1} & 2^n - n\,2^{n-1} \end{pmatrix}.$$

[7]

Question 10

The Fibonacci numbers $F_1, F_2, F_3, \ldots$ are defined by

$$F_1 = 1, \qquad F_2 = 1, \qquad F_{r+2} = F_{r+1} + F_r \quad \text{for } r \geq 1.$$

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Find $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$, and verify that each agrees with the matrix $\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$, where F_0 is taken to be 0. [3]

(b) Prove by mathematical induction that

$$\mathbf{A}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

for all integers $n \geq 2$. [6]

SECTION C • QUESTIONS 11–15 • Comparisons

Question 11

Prove by mathematical induction that $2^n > 2n$ for all integers $n \geq 3$.

State why the result is not true for $n = 2$.

[6]

Question 12

Prove by mathematical induction that $n! > 2^n$ for all integers $n \geq 4$.

[6]

Question 13

Prove by mathematical induction that $4^n > n^4$ for all integers $n \geq 5$.

State the value of n for which $4^n = n^4$.

[7]

Question 14

The sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 1, \qquad u_{n+1} = \sqrt{2 + u_n} \quad \text{for } n \geq 1.$$

(a) Prove by mathematical induction that $u_n < 2$ for all positive integers n. [6]

(b) Hence show that $u_{n+1} > u_n$ for all positive integers n. [3]

Question 15

Prove by mathematical induction that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > \sqrt{n}$$

for all integers $n \geq 2$.

[7]

SECTION D • QUESTIONS 16–20 • Divisibility

Question 16

Prove by mathematical induction that $6^n - 1$ is divisible by 5 for all positive integers n.

[5]

Question 17

Prove by mathematical induction that $n^3 + 11n$ is divisible by 6 for all positive integers n.

[6]

Question 18

Prove by mathematical induction that $4^n + 6n - 1$ is divisible by 9 for all positive integers n.

[6]

Question 19

Prove by mathematical induction that $3^{2n+1} + 2^{n+2}$ is divisible by 7 for all positive integers n.

[6]

Question 20

Prove by mathematical induction that $11^{n+1} + 12^{2n-1}$ is divisible by 133 for all positive integers n.

[7]

SECTION E • QUESTIONS 21–25 • Derivatives

Question 21

Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{x+2}\right) = \frac{(-1)^n\, n!}{(x+2)^{n+1}}.$$

[6]

Question 22

Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^3\mathrm{e}^x\right) = \Big(x^3 + 3nx^2 + 3n(n-1)x + n(n-1)(n-2)\Big)\mathrm{e}^x.$$

[7]

Question 23

(a) Express $\dfrac{1}{x^2-1}$ in partial fractions. [2]

(b) Hence prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{x^2-1}\right) = \frac{(-1)^n\, n!}{2}\left[\frac{1}{(x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right].$$

[6]

Question 24

It is given that $y = \sin x \cos x$.

(a) Show that $y = \frac{1}{2}\sin 2x$. [1]

(b) Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^{2n}y}{\mathrm{d}x^{2n}} = (-1)^n\, 2^{2n-1}\sin 2x.$$

[7]

Question 25

It is given that $y = x\cos 2x$.

(a) Show that $\dfrac{d^2y}{dx^2} = -4\left(x\cos 2x + \sin 2x\right)$. [2]

(b) Prove by mathematical induction that, for all positive integers n,

$$\frac{d^{2n}y}{dx^{2n}} = (-1)^n\, 2^{2n}\left(x\cos 2x + n\sin 2x\right).$$

[7]

SECTION F • QUESTIONS 26–32 • Mixed

Question 26

The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 2$ and $u_{n+1} = 5u_n - 4$ for $n \geq 1$.

Prove by mathematical induction that $u_n = 5^{n-1} + 1$ for all positive integers n.

[5]

Question 27

Prove by mathematical induction that

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{n}\right) = n+1$$

for all positive integers n.

[6]

Question 28

A finite set S has n elements, where $n \geq 1$.

Prove by mathematical induction that S has exactly 2^n subsets, including the empty set and S itself.

[6]

Question 29

A puzzle consists of n discs of different sizes stacked on one of three pegs, with each disc resting on a larger one. The discs are to be moved one at a time onto another peg, keeping them in the same order, and a disc may never be placed on a smaller disc. Let M_n be the least number of moves needed.

(a) Write down the values of M_1 and M_2. [2]

(b) Explain why $M_{n+1} = 2M_n + 1$. [2]

(c) Prove by mathematical induction that $M_n = 2^n - 1$ for all positive integers n. [5]

Question 30

A number of straight lines are drawn in a plane, no two of them parallel and no three of them passing through the same point. Let R_n be the number of regions into which n such lines divide the plane.

(a) Write down the values of R_1, R_2 and R_3. [2]

(b) Explain why $R_{k+1} = R_k + k + 1$. [2]

(c) Conjecture a formula for R_n and prove it by mathematical induction. [6]

Question 31

A student is investigating the statement

$$P(n): \qquad n^2 + n \text{ is odd.}$$

(a) Show that if $P(k)$ is true then $P(k+1)$ is true. [3]

(b) Show that $n^2 + n$ is in fact *even* for every positive integer n. [2]

(c) Explain carefully why parts (a) and (b) do not contradict one another, and state what this shows about the role of the base case in a proof by induction. [3]

Question 32

The sequence $u_1, u_2, u_3, \ldots$ is such that

$$u_1 = 1, \qquad u_2 = 5, \qquad u_{n+2} = 5u_{n+1} - 6u_n \quad \text{for } n \geq 1.$$

(a) Find u_3 and u_4. [2]

(b) Prove by mathematical induction that $u_n = 3^n - 2^n$ for all positive integers n. [7]

The recurrence uses the two previous terms, so this proof needs two base cases and an inductive step that assumes the result for two consecutive values.