Further Math with Ali Hashir

Proof by Induction

Further Pure Mathematics 1 – Chapter 7 • Answers

A note on this booklet. Most questions in this chapter ask for a *proof*, and a proof has no short answer — the working *is* the answer. Where a question also asks for something that can be written down (a value, a matrix, a conjecture, an explanation), it is given below; otherwise the entry simply points you to the *Worked Solutions* booklet, where every proof is set out in full.

Use this booklet to check the parts you can check, then compare your proof with the worked one — looking in particular at whether you stated the assumption, and whether you wrote the concluding sentence.

SECTION A • QUESTIONS 1–5 • Summations

1 Proof — see the *Worked Solutions* booklet.

2 Proof — see the *Worked Solutions* booklet.

3 **(a)** Proof — see the *Worked Solutions* booklet. **(b)** $\frac{1}{3}$

4 Proof — see the *Worked Solutions* booklet.

5 Proof — see the *Worked Solutions* booklet.

SECTION B • QUESTIONS 6–10 • Matrices

6 Proof — see the *Worked Solutions* booklet.

7 **(a)** Proof — see the *Worked Solutions* booklet. **(b)** $(\mathbf{A}^n)^{-1} = \begin{pmatrix} 1 & 3^{-n}-1 \\ 0 & 3^{-n} \end{pmatrix}$

8 **(a)** $\mathbf{B}^2 = \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix}$, $\mathbf{B}^3 = \begin{pmatrix} 8 & 12 \\ 0 & 8 \end{pmatrix}$ **(b)** Proof — see the *Worked Solutions* booklet.

9 Proof — see the *Worked Solutions* booklet.

10 **(a)** $\mathbf{A}^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$, $\mathbf{A}^4 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ **(b)** Proof — see the *Worked Solutions* booklet.

SECTION C • QUESTIONS 11–15 • Comparisons

11 Proof — see the *Worked Solutions* booklet. At $n = 2$, 2^n and $2n$ are *equal* (both 4), so the strict inequality fails.

12 Proof — see the *Worked Solutions* booklet.

13 Proof — see the *Worked Solutions* booklet. $4^n = n^4$ when $n = 4$.

14 **(a)** Proof — see the *Worked Solutions* booklet. **(b)** $u_{n+1}^2 - u_n^2 = -(u_n - 2)(u_n + 1) > 0$

15 Proof — see the *Worked Solutions* booklet.

SECTION D • QUESTIONS 16–20 • Divisibility

16 Proof — see the *Worked Solutions* booklet.

17 Proof — see the *Worked Solutions* booklet.

18 Proof — see the *Worked Solutions* booklet.

19 Proof — see the *Worked Solutions* booklet.

20 Proof — see the *Worked Solutions* booklet.

SECTION E • QUESTIONS 21–25 • Derivatives

21 Proof — see the *Worked Solutions* booklet.

22 Proof — see the *Worked Solutions* booklet.

23 **(a)** $\dfrac{1}{x^2-1} = \dfrac{1}{2}\left[\dfrac{1}{x-1} - \dfrac{1}{x+1}\right]$ **(b)** Proof — see the *Worked Solutions* booklet.

24 **(a)** $\sin x \cos x = \frac{1}{2}\sin 2x$ **(b)** Proof — see the *Worked Solutions* booklet.

25 **(a)** $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -4\,(x\cos 2x + \sin 2x)$ **(b)** Proof — see the *Worked Solutions* booklet.

SECTION F • QUESTIONS 26–32 • Mixed

26 Proof — see the *Worked Solutions* booklet.

27 Proof — see the *Worked Solutions* booklet.

28 Proof — see the *Worked Solutions* booklet.

29 **(a)** $M_1 = 1,\ M_2 = 3$ **(b)** move the top n discs to the spare peg, then the largest, then the n back **(c)** Proof — see the *Worked Solutions* booklet.

30 **(a)** $R_1 = 2,\ R_2 = 4,\ R_3 = 7$ **(b)** the new line is cut into $k+1$ pieces, each splitting one region in two **(c)** $R_n = \frac{1}{2}\left(n^2+n+2\right)$

31 **(a)** $(k+1)^2 + (k+1) = \left(k^2+k\right) + 2\,(k+1)$, and adding an even number preserves oddness **(b)** $n^2 + n = n\,(n+1)$, a product of consecutive integers **(c)** the implication is true but no base case exists, so the chain never starts

32 **(a)** $u_3 = 19,\ u_4 = 65$ **(b)** Proof — see the *Worked Solutions* booklet.

Step-by-step working for every question is in the companion Worked Solutions booklet.