Further Math with Ali Hashir

Polar Coordinates

Further Pure Mathematics 1 – Chapter 5 • CIE 9231

About this chapter

Cartesian coordinates fix a point by how far across and how far up it is. Polar coordinates fix it by how far away it is and in what direction: the distance r from a fixed point O, called the *pole*, and the angle θ turned anticlockwise from a fixed ray out of O, called the *initial line*. Nothing has changed about the plane — only the way we label it.

That relabelling makes some curves far easier to describe. A circle centred at the pole is simply $r = a$; a line through the pole is $\theta = \alpha$; and spirals, cardioids and the many-petalled roses, all of them awkward in Cartesian form, become short equations in r and θ. Half of this chapter is learning to move between the two descriptions, and learning which shape to expect from which kind of equation.

The other half is area. A region measured from the pole is naturally cut into thin *sectors* rather than thin rectangles, and that is where $A = \frac{1}{2}\int r^2\,d\theta$ comes from. Almost every question in the second half of this worksheet is that one formula, applied to a curve you must first understand well enough to sketch — because the sketch is what tells you the limits.

Key results: coordinates and equations

1. The convention. Throughout 9231, $r \geq 0$. So if an equation gives a negative value of r (or a negative r^2) at some θ, the curve simply *has no point* there: that value of θ is outside its domain. Angles are measured anticlockwise from the initial line.

2. Converting a point.

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r = \sqrt{x^2+y^2}, \qquad \tan\theta = \frac{y}{x}.$$

Always choose θ from the *quadrant* the point lies in; a calculator's $\tan^{-1}$ will often give you the wrong one.

3. Converting an equation. Going *polar to Cartesian*, look for the combinations $r\cos\theta$, $r\sin\theta$ and r^2, and replace them by x, y and x^2+y^2. Multiplying through by r (or by r^2) is usually what creates them, and it never loses the pole. Going *Cartesian to polar*, substitute $x = r\cos\theta$ and $y = r\sin\theta$ and simplify.

4. Curves worth recognising.

$r = a$	circle of radius a centred at the pole
$\theta = \alpha$	half-line from the pole at angle α
$r = 2a\cos\theta$	circle of radius a through the pole, centre $(a, 0)$
$r = a\sec\theta$	the line $x = a$
$r = a(1 \pm \cos\theta)$, $r = a(1 \pm \sin\theta)$	cardioid (heart-shaped, passing through the pole)
$r = a + b\cos\theta$	limaçon: an oval if $a > b$, dimpled or looped if $a < b$
$r = a\cos n\theta$, $r = a\sin n\theta$	rose with petals of length a
$r = a\theta$, $r = ae^{k\theta}$	spirals

Key results: sketching and area

5. How to sketch. Detailed plotting is never required, but the significant features are. Find, in this order: the values of θ for which $r = 0$ (where the curve passes through the pole); the greatest and least values of r and where they occur; the value of r at $\theta = 0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi$; and the domain, if $r \geq 0$ restricts it.

6. Symmetry halves the work. If replacing θ by $-\theta$ leaves the equation unchanged, the curve is symmetrical about the *initial line*. If replacing θ by $\pi - \theta$ leaves it unchanged, it is symmetrical about the half-line $\theta = \frac{1}{2}\pi$.

7. Area of a sector. The region swept out from the pole between the half-lines $\theta = \alpha$ and $\theta = \beta$, bounded by the arc of $r = \mathrm{f}(\theta)$, has area

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, \mathrm{d}\theta.$$

Note that it is r^2, not r, that is integrated — so an equation given in the form $r^2 = \mathrm{f}(\theta)$ can be substituted in with no algebra at all.

8. Two curves. To find where they meet, solve $r_1 = r_2$ — and then check the pole *separately*, since both curves may pass through it at different values of θ, which no such equation can detect. For a region inside both curves, the boundary at each θ is whichever curve is *nearer* the pole; split the integral at the intersection.

9. Greatest distances. The distance of a point of the curve from the initial line is $y = r\sin\theta$, and its distance from the half-line $\theta = \frac{1}{2}\pi$ is $x = r\cos\theta$. Maximise whichever is asked for by differentiating with respect to θ. For the greatest distance from the *pole*, maximise r — or, more easily, maximise r^2.

10. The two identities you will use constantly.

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta), \qquad \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta).$$

Squaring r almost always produces a $\cos^2$ or $\sin^2$ term, and it cannot be integrated until one of these has been used.

How to use this worksheet

- The twenty questions are in **increasing order of difficulty**: 1–7 cover coordinates, conversion and first areas; 8–14 build up sketching, areas and greatest distances; 15–20 are examination-length. Work through them in order.
- Marks are in square brackets. Allow about **$1\frac{1}{2}$ minutes per mark**; the full worksheet is roughly **4 hours** of work.
- **Sketch first, integrate second.** The sketch is what tells you the limits of the integral and which curve bounds the region. A neat sketch showing the pole, the greatest and least r, and any symmetry is worth more than a careful curve drawn from a table of values.
- Remember $r \geq 0$. If an equation restricts the domain of θ, say so *and say why* — it is usually the first few marks of the question.
- Write down $\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, \mathrm{d}\theta$ with its limits *before* you start integrating, and keep the $\frac{1}{2}$ in front where you can see it.
- Where a question says *show that*, all the marks are in the working. For a “verify that there is a root between a and b”, evaluate at both ends, state that the sign changes, and say that the function is continuous.
- Leave every answer **exact** unless a number of decimal places or significant figures is asked for.

QUESTIONS 1–7 • Coordinates, conversion and first areas

Question 1

(a) Find the Cartesian coordinates of the point with polar coordinates $\left(4, \frac{2}{3}\pi\right)$. [2]

(b) Find the polar coordinates (r, θ), where $r \geq 0$ and $0 \leq \theta < 2\pi$, of each of the points with Cartesian coordinates $(-3, 3)$, $(0, -5)$ and $\left(1, -\sqrt{3}\right)$. [5]

Question 2

Find a polar equation for each of the following curves, giving your answer in the form $r = \mathrm{f}(\theta)$ where possible.

(a) $x^2 + y^2 = 25$ [1]

(b) $y = \sqrt{3}\,x$, for $x \geq 0$ [2]

(c) $x^2 + y^2 = 6y$ [2]

(d) $2x - 5y = 4$ [2]

Question 3

Find a Cartesian equation for each of the following curves.

(a) $r = 5\sec\theta$ [2]

(b) $r = 4\cos\theta + 2\sin\theta$, and hence show that this curve is a circle, stating its centre and radius. [5]

(c) $r\,(1 + 2\cos\theta) = 3$ [4]

Question 4

The curve C has polar equation $r = 3 + \cos\theta$, for $0 \le \theta < 2\pi$.

(a) Write down the value of r when $\theta = 0$, $\frac{1}{2}\pi$, π and $\frac{3}{2}\pi$. [2]

(b) State the greatest and least distances of a point of C from the pole, and the value of θ at which each occurs. [2]

(c) Sketch C, and state the equation of its line of symmetry. [3]

Question 5

The curve C has polar equation $r = 1 + \cos 2\theta$, for $0 \le \theta < 2\pi$.

(a) State the greatest and least values of r, and the values of θ at which C passes through the pole. [2]

(b) Sketch C. [2]

(c) Find the exact total area enclosed by C. [4]

Question 6

The curve C has polar equation $r = \theta^2$, for $0 \le \theta \le \pi$.

(a) Sketch C, stating the polar coordinates of the point of C furthest from the pole. [2]

(b) Find the exact area of the region bounded by C, the initial line and the half-line $\theta = \pi$. [3]

(c) Find the exact area of the part of that region lying between the half-lines $\theta = \frac{1}{2}\pi$ and $\theta = \pi$. [2]

Question 7

The curve C has polar equation $r = 6\cos\theta$, for $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$.

(a) Show that C is a circle, and state its centre and radius in Cartesian form. [3]

(b) Sketch C. [2]

(c) Find the exact area of the region bounded by C, the initial line and the half-line $\theta = \frac{1}{3}\pi$. [4]

QUESTIONS 8–14 • Sketching, areas and greatest distances

Question 8

The straight line l has Cartesian equation $x + y = 4$.

(a) Show that l has polar equation $r = \dfrac{4}{\cos\theta + \sin\theta}$, and deduce that this may be written as $r = 2\sqrt{2}\,\sec\left(\theta - \frac{1}{4}\pi\right)$. [4]

(b) Hence write down the least distance of a point of l from the pole, and the value of θ at which it occurs. [2]

(c) The region R is bounded by l, the initial line and the half-line $\theta = \frac{1}{2}\pi$. Use the polar area formula to show that R has area 8, and confirm this using the Cartesian equation. [5]

Question 9

The curve C has polar equation $r = 3 - 2\cos\theta$, for $0 \le \theta < 2\pi$.

(a) State the greatest and least distances of a point of C from the pole. [2]

(b) Sketch C, and state the equation of its line of symmetry. [2]

(c) Find the exact total area enclosed by C. [4]

Question 10

The curve C has polar equation $r = 3\cos^2\theta$, for $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$.

(a) Sketch C, and state the equation of its line of symmetry. [2]

(b) Show that C has Cartesian equation $\left(x^2+y^2\right)^{\frac{3}{2}} = 3x^2$. [3]

(c) Find the exact area of the region enclosed by C. [4]

(d) Find the exact greatest distance of a point of C from the initial line. [5]

Question 11

The curve C has polar equation $r = \dfrac{1}{2-\cos\theta}$, for $0 \le \theta < 2\pi$.

(a) State the greatest and least distances of a point of C from the pole, and the values of θ at which they occur. [2]

(b) Show that C has Cartesian equation $3x^2 + 4y^2 - 2x = 1$. [4]

(c) Hence, by completing the square, find the Cartesian coordinates of the centre of C. [3]

Question 12

The curves C_1 and C_2 have polar equations

$$C_1: \ r = 4\cos\theta, \qquad C_2: \ r = 4\sin\theta, \qquad \text{for } 0 \le \theta \le \tfrac{1}{2}\pi.$$

(a) Show that each curve is an arc of a circle, stating the centre and radius of each. [3]

(b) Find the polar coordinates of the point of intersection of C_1 and C_2 other than the pole. [2]

(c) Sketch C_1 and C_2 on a single diagram. [2]

(d) Find the exact area of the region lying inside both curves. [5]

Question 13

The curve C has polar equation $r^2 = \theta \sin\theta$, for $0 \le \theta \le \pi$.

(a) Show that, at the point of C furthest from the pole,

$$\tan\theta + \theta = 0,$$

and verify that this equation has a root between 2.0 and 2.1. [5]

(b) Sketch C. [2]

(c) Find the exact area of the region enclosed by C. [4]

Question 14

The curve C has polar equation

$$r^2 = \frac{\theta}{1+\theta^2}, \qquad 0 \le \theta \le 2.$$

(a) Show that the greatest distance of a point of C from the pole is $\frac{1}{\sqrt{2}}$, and state the value of θ at which it occurs. [4]

(b) Sketch C. [2]

(c) Find the exact area of the region bounded by C and the half-line $\theta = 2$. [4]

QUESTIONS 15–20 • Examination-standard problems

Question 15

The curves C_1 and C_2 have polar equations

$$C_1: \ r = 3\sin\theta, \qquad C_2: \ r = 1+\sin\theta, \qquad \text{for } 0 \le \theta \le \pi.$$

(a) Show that C_1 is an arc of a circle, stating its centre and radius. [2]

(b) Find the polar coordinates of the two points, other than the pole, at which C_1 and C_2 meet. [3]

(c) Sketch C_1 and C_2 on a single diagram. [3]

(d) Find the exact area of the region lying inside both curves. [6]

Question 16

The curve C has polar equation $r = 1 + e^{-\theta}$, for $0 \leq \theta \leq \pi$.

(a) Sketch C, stating the polar coordinates of its endpoints. [3]

(b) Find the exact area of the region bounded by C, the initial line and the half-line $\theta = \pi$. [5]

(c) Show that, at the point of C furthest from the initial line,

$$\left(1 + e^{-\theta}\right) \cos\theta - e^{-\theta} \sin\theta = 0,$$

and verify that this equation has a root between 1.3 and 1.4. [5]

Question 17

The curve C has polar equation $r = e^{\theta}$, for $0 \leq \theta \leq \ln 3$.

(a) Sketch C, stating the polar coordinates of its endpoints. [3]

(b) The region R is bounded by C, the initial line and the half-line $\theta = \ln 3$. Show that R has area 2. [3]

(c) The half-line $\theta = \alpha$, where $0 < \alpha < \ln 3$, divides R into two regions of equal area. Find the exact value of α. [4]

Question 18

The curve C has polar equation

$$r = \frac{1}{1 + \sin\theta}, \qquad -\tfrac{1}{2}\pi < \theta < \tfrac{3}{2}\pi.$$

(a) Show that C has Cartesian equation $x^2 = 1 - 2y$, and hence sketch C. [4]

(b) State the least distance of a point of C from the pole, and the value of θ at which it occurs. [2]

(c) Show that $(1 + \sin\theta)^2 = 4\cos^4\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\theta\right)$. [3]

(d) Hence find the exact area of the region bounded by C, the initial line and the half-line $\theta = \tfrac{1}{2}\pi$. [5]

Question 19

The curve C has polar equation $r = 1 + 2\cos\theta$, where $r \geq 0$.

(a) Show that C is defined only for $-\frac{2}{3}\pi \leq \theta \leq \frac{2}{3}\pi$, and sketch C. [4]

(b) Find the exact area of the region enclosed by C. [5]

(c) Show that, at the point of C furthest from the initial line,

$$4\cos^2\theta + \cos\theta - 2 = 0,$$

and hence find that greatest distance, correct to 3 significant figures. [6]

Question 20

The curve C has polar equation $r = 2\sin\theta\tan\theta$, for $0 \leq \theta < \frac{1}{2}\pi$.

(a) Show that C has Cartesian equation $y^2(2 - x) = x^3$, and write down the equation of the asymptote of C. [5]

(b) Sketch the part of C for which $0 \leq \theta \leq \frac{1}{4}\pi$, stating the polar coordinates of its endpoint. [3]

(c) Show that $r^2 = 4\left(\sec^2\theta - 2 + \cos^2\theta\right)$. [3]

(d) Hence find the exact area of the region bounded by C and the half-line $\theta = \frac{1}{4}\pi$. [5]