Further Math with Ali Hashir

Matrices

Further Pure Mathematics 1 – Chapter 4 • CIE 9231

About this chapter

A matrix is two things at once, and this chapter only makes sense if you hold on to both. It is an array of numbers with an arithmetic of its own; and it is a *machine* that takes each point of the plane to another point. The first view gives determinants, inverses and matrix equations. The second gives rotations, reflections, stretches and shears — and the question of what a transformation leaves alone.

The determinant is what ties the two views together. For a 2×2 matrix it is a single number that settles everything: it is zero exactly when the matrix has no inverse, which is exactly when the transformation flattens the whole plane onto a line and so cannot be undone; and when it is not zero, its size is the factor by which every area is multiplied and its sign says whether the plane has been turned over. Nothing else in Further Pure 1 does so much work.

The last idea of the chapter is *invariance*. A point is invariant if the transformation leaves it exactly where it was. A line is invariant if every point of it is sent to a point of the same line — not necessarily to itself, so the points may slide along it. Those are different conditions, and separating them is where most of the marks in an examination question live.

Key results: the arithmetic

1. Multiplication. A product $(m\times n)(n\times p)$ exists only when the inner numbers match, and the result is $m\times p$. Multiplication is associative, $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$, but it is *not* commutative: in general $\mathbf{AB}\neq\mathbf{BA}$.

2. Determinant and inverse of a 2×2 matrix. For $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$,

$$\det\mathbf{M} = ad-bc, \qquad \mathbf{M}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a\end{pmatrix} \quad \text{provided } \det\mathbf{M}\neq 0.$$

$\mathbf{M}$ is *singular* $\iff \det\mathbf{M} = 0 \iff \mathbf{M}^{-1}$ does not exist.

3. Determinant of a 3×3 matrix. Expand along any row or column, using the sign pattern $\left(\begin{smallmatrix} + & - & + \\ - & + & - \\ + & - & +\end{smallmatrix}\right)$ and the 2×2 determinant left when that entry's row and column are deleted. Choosing a row or column that contains a zero saves a whole 2×2 determinant.

4. Inverse of a 3×3 matrix, by row operations. Write the matrix and the identity side by side and reduce the left-hand block to $\mathbf{I}$:

$$(\mathbf{M}\,|\,\mathbf{I}) \longrightarrow \cdots \longrightarrow (\mathbf{I}\,|\,\mathbf{M}^{-1}).$$

The three permitted row operations are: *swap* two rows; *multiply* a row by a non-zero constant; *add a multiple* of one row to another. Apply each one to the whole row, so that the right-hand block changes with the left. Work down the columns to clear the entries below each leading 1, then work back up to clear those above. If a row of the left-hand block ever becomes entirely zero, $\mathbf{M}$ is singular and has no inverse.

5. Products. $\det(\mathbf{AB}) = \det\mathbf{A}\cdot\det\mathbf{B}$, and the inverse of a product reverses the order: $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Key results: transformations and invariance

6. The columns are the images of the unit vectors. Column 1 is the image of $\binom{1}{0}$ and column 2 is the image of $\binom{0}{1}$; between them they pin the transformation down completely. In particular:

rotation through θ anticlockwise about O	$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$
reflection in the line $y = mx$	$\dfrac{1}{1+m^2}\begin{pmatrix}1-m^2 & 2m\\ 2m & m^2-1\end{pmatrix}$
enlargement, centre O, scale factor k	$k\mathbf{I} = \begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}$
stretch parallel to the y-axis, factor k (x-axis fixed)	$\begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix}$
shear, x-axis fixed, with $(0,1) \mapsto (k,1)$	$\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}$
shear, y-axis fixed, with $(1,0) \mapsto (1,k)$	$\begin{pmatrix}1 & 0\\ k & 1\end{pmatrix}$

7. Sequences of transformations. $\mathbf{AB}$ represents "$\mathbf{B}$ first, then $\mathbf{A}$": **the matrix applied first goes on the right.** The transformation represented by $\mathbf{M}^{-1}$ undoes the one represented by $\mathbf{M}$.

8. Area. Under the transformation represented by $\mathbf{M}$,

$$\text{area of image} = |\det \mathbf{M}| \times \text{area of object},$$

and a negative determinant means the plane has been turned over. The image of the unit square is the parallelogram spanned by the two columns of $\mathbf{M}$, so its area is $|\det \mathbf{M}|$.

9. Invariant points satisfy $\mathbf{Mv} = \mathbf{v}$, that is $(\mathbf{M} - \mathbf{I})\mathbf{v} = \mathbf{0}$. The origin always is one. There is a whole *line* of invariant points precisely when $\det(\mathbf{M} - \mathbf{I}) = 0$; otherwise the origin is the only one.

10. Invariant lines through the origin are found by substituting $y = mx$: take the point $\binom{x}{mx}$, work out its image, and insist that the image satisfies the same equation for every x. That gives an equation in m, usually a quadratic, whose roots are the gradients of the invariant lines. The line $x = 0$ has no gradient, so it can never appear among those roots and must always be tested separately.

How to use this worksheet

- The twenty questions are in **increasing order of difficulty**: 1–7 rehearse the arithmetic, 8–14 build up transformations and invariance, 15–20 are examination-length. Work through them in order.
- Marks are in square brackets. Allow about **$1\frac{1}{2}$ minutes per mark**; the full worksheet is roughly **$2\frac{1}{2}$ hours** of work.
- **Show the multiplication.** In an inverse question the row operations and the determinant carry the marks, not the final array. Name each operation as you use it, for instance $R_2 \to R_2 - 2R_1$.
- *Give full details* means the type of transformation **and** whatever pins it down: the centre and angle of a rotation, the mirror line of a reflection, the fixed line together with the direction and scale factor of a stretch, the fixed line together with the image of one point for a shear.
- For a sequence of transformations, **the one applied first goes on the right**. If you are unsure, test your matrix on a single point and follow that point through the sequence by hand.
- Keep invariant *points* and invariant *lines* apart, and check the line $x = 0$ separately every time.
- Leave every answer **exact** — fractions and surds, never decimals.

QUESTIONS 1–7 • Matrix arithmetic, determinants and inverses

Question 1

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 5 \\ -2 & 1 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 4 \end{pmatrix}.$$

(a) Find $2\mathbf{A} - 3\mathbf{B}$. [2]

(b) Find $\mathbf{AB}$ and $\mathbf{BA}$, and state what your answers show. [4]

(c) Find $\mathbf{AC}$, and explain why $\mathbf{CA}$ does not exist. [3]

Question 2

(a) Evaluate the determinant of $\begin{pmatrix} 5 & -2 \\ 3 & 4 \end{pmatrix}$. [1]

(b) Show that $\begin{pmatrix} 6 & -9 \\ -4 & 6 \end{pmatrix}$ is singular, and state what this tells you about the matrix. [2]

(c) The matrix $\mathbf{M} = \begin{pmatrix} k-1 & 2 \\ 3 & k-2 \end{pmatrix}$ is singular. Find the two possible values of k. [4]

Question 3

(a) Evaluate $\det \mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 5 & 2 & -2 \end{pmatrix}$. [3]

(b) The matrix $\mathbf{N} = \begin{pmatrix} k & 3 & 0 \\ 1 & k & 2 \\ 0 & 1 & 1 \end{pmatrix}$ is singular. Find the possible values of k. [4]

Question 4

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}.$$

(a) Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$. [4]

(b) Find the matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$. [3]

(c) Find the matrix $\mathbf{Y}$ such that $\mathbf{YA} = \mathbf{B}$, and state why $\mathbf{Y} \neq \mathbf{X}$. [3]

Question 5

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 1 \end{pmatrix}$.

(a) Show that $\mathbf{A}$ is non-singular. [2]

(b) Use row operations to find $\mathbf{A}^{-1}$. [6]

(c) Verify your answer by evaluating the first column of $\mathbf{AA}^{-1}$. [2]

Question 6

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}.$$

(a) Find $\mathbf{AB}$ and hence $(\mathbf{AB})^{-1}$. [4]

(b) Find $\mathbf{B}^{-1}\mathbf{A}^{-1}$ and verify that it equals $(\mathbf{AB})^{-1}$. [4]

(c) The matrices $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are non-singular and $\mathbf{PQR} = \mathbf{I}$. Express $\mathbf{Q}$ in terms of $\mathbf{P}$ and $\mathbf{R}$. [2]

Question 7

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$.

(a) Find $\mathbf{A}^2$ and show that $\mathbf{A}^2 - 6\mathbf{A} + 5\mathbf{I} = \mathbf{0}$. [3]

(b) By multiplying this equation by $\mathbf{A}^{-1}$, express $\mathbf{A}^{-1}$ in the form $p\mathbf{A} + q\mathbf{I}$ and hence write down $\mathbf{A}^{-1}$. [4]

(c) Show that $\mathbf{A}^3 = 31\mathbf{A} - 30\mathbf{I}$. [3]

QUESTIONS 8–14 • Transformations, areas and invariance

Question 8

Give full details of the geometrical transformation in the x–y plane represented by each of the following matrices.

(a) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ [2]

(b) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ [2]

(c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ [2]

(d) $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$ [2]

(e) $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ [2]

Question 9

Three transformations of the x–y plane are defined as follows.

T_1: reflection in the x-axis
T_2: rotation through $90°$ anticlockwise about the origin
T_3: stretch parallel to the x-axis, scale factor 2

The single matrix $\mathbf{M}$ represents T_1 followed by T_2 followed by T_3.

(a) Write down the matrix representing each of T_1, T_2 and T_3. [3]

(b) Show that $\mathbf{M} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$. [3]

(c) The triangle with vertices $(1, 1)$, $(3, 1)$ and $(2, 4)$ is transformed by $\mathbf{M}$. Find the vertices of the image, and the area of the image. [4]

(d) Find $\mathbf{M}^{-1}$. [2]

Question 10

(a) The triangle PQR has vertices $P(0, 0)$, $Q(4, 1)$ and $R(2, 5)$. It is transformed by the matrix $\begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$ onto the triangle $P'Q'R'$. Find the area of $P'Q'R'$. [4]

(b) A region of area $6\,\text{cm}^2$ is transformed by $\begin{pmatrix} k & 3 \\ 2 & k \end{pmatrix}$ onto a region of area $60\,\text{cm}^2$. Find the possible values of k. [4]

(c) The transformation represented by $\begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$ maps a quadrilateral S onto a quadrilateral of area $45\,\text{cm}^2$. Find the area of S. [2]

Question 11

The unit square $OABC$ has vertices $O(0,0)$, $A(1,0)$, $B(1,1)$ and $C(0,1)$. It is transformed by $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ onto the parallelogram $OA'B'C'$.

(a) Find the coordinates of A', B' and C'. [3]

(b) Write down the area of $OA'B'C'$, and find the matrix which transforms $OA'B'C'$ back onto $OABC$. [3]

(c) A different transformation maps the unit square $OABC$ onto the parallelogram with vertices $O(0,0)$, $(2,-1)$, $(5,2)$ and $(3,3)$. Find the matrix of this transformation, and the area of the parallelogram. [4]

Question 12

(a) Show that the transformation represented by $\begin{pmatrix} 5 & -2 \\ 6 & -2 \end{pmatrix}$ has a line of invariant points, and find its equation. [4]

(b) Show that the origin is the only invariant point of the transformation represented by $\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$. [3]

(c) The transformation represented by $\begin{pmatrix} k & 4 \\ 1 & k \end{pmatrix}$ has a line of invariant points. Find the two possible values of k, and the corresponding lines. [5]

Question 13

(a) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$. [5]

(b) Find the equations of the invariant lines, through the origin, of the transformation represented by $\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$. [4]

(c) Show that the transformation represented by $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ has exactly one invariant line through the origin, and state its equation. [4]

Question 14

The matrix $\mathbf{M}$ represents the sequence of two transformations in the x–y plane given by a stretch parallel to the y-axis with scale factor 2, followed by a shear with the x-axis fixed in which $(0, 1)$ is mapped to $(3, 1)$.

(a) Show that $\mathbf{M} = \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$. [4]

(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is $\mathbf{I}$. [2]

(c) Find the equations of the invariant lines, through the origin, of the transformation represented by $\mathbf{M}$. [5]

(d) State which of these lines is a line of invariant points, and explain why it must be. [2]

QUESTIONS 15–20 • Examination-standard problems

Question 15

The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} -1 & k & 1 \\ 0 & 1 & 2 \\ 1 & -2 & 1 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 2 & -1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 0 \end{pmatrix},$$

where k is a real constant.

(a) Given that $\mathbf{A}$ is singular, find the value of k. [4]

(b) Using this value of k, show that $\mathbf{CAB} = \begin{pmatrix} 7 & -6 \\ -3 & 4 \end{pmatrix}$. [4]

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{CAB}$. [5]

Question 16

The matrix $\mathbf{M}$ represents a reflection in the x-axis followed by a rotation through angle θ anticlockwise about the origin, where $0 < \theta < \pi$.

(a) Show that $\mathbf{M} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, and write down $\det \mathbf{M}$. [4]

(b) Show that the invariant lines of the transformation through the origin have gradients $\dfrac{1-\cos\theta}{\sin\theta}$ and $-\dfrac{1+\cos\theta}{\sin\theta}$, and deduce that these two lines are perpendicular. [6]

(c) Given that $\theta = \frac{2}{3}\pi$, find $\mathbf{M}$ and the equations of its two invariant lines. [3]

(d) Show that $\mathbf{M}^2 = \mathbf{I}$ for every θ, and explain geometrically what this tells you about the transformation. [3]

Question 17

The transformation T in the x–y plane is represented by $\mathbf{M} = \begin{pmatrix} 5 & k \\ 2 & 1 \end{pmatrix}$, where k is a constant.

(a) Show that the line $y = mx$ is invariant under T if and only if $km^2 + 4m - 2 = 0$. [3]

(b) Hence find the set of values of k for which T has two distinct invariant lines of the form $y = mx$, the value of k for which it has exactly one, and the set of values for which it has none. [5]

(c) Show that in the case $k = 0$ the transformation nevertheless has two invariant lines through the origin, and state their equations. [4]

Question 18

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & k & 2 \\ 3 & -1 & 1 \end{pmatrix}$, where k is a real constant.

(a) Show that $\det \mathbf{A} = 5(k + 2)$, and hence state the set of values of k for which $\mathbf{A}$ is non-singular. [4]

(b) For the case $k = 0$, find $\mathbf{A}^{-1}$. [6]

(c) Hence, still with $k = 0$, find the column vector $\mathbf{x}$ for which $\mathbf{Ax} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}$. [3]

Question 19

The matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has real entries and satisfies $\mathbf{M}^2 = \mathbf{I}$, where $\mathbf{M} \neq \mathbf{I}$ and $\mathbf{M} \neq -\mathbf{I}$.

(a) Show that $a + d = 0$, and deduce that $\det \mathbf{M} = -1$. [6]

(b) It is given further that $\mathbf{M} = \frac{1}{5}\begin{pmatrix} 3 & p \\ 4 & q \end{pmatrix}$, where p and q are constants. Find p and q. [3]

(c) Show that the transformation represented by this $\mathbf{M}$ is a reflection, and find the equation of the mirror line. [4]

(d) Write down the image of the point $(5, 0)$ under this transformation. [2]

Question 20

A transformation of the x–y plane is represented by $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $b \neq 0$.

(a) Show that the line $y = mx$ is invariant under the transformation if and only if

$$bm^2 + (a - d)m - c = 0,$$

and explain why, when $b \neq 0$, the line $x = 0$ cannot be invariant. [5]

(b) Hence write down, in terms of a, b, c and d, the condition for the transformation to have exactly one invariant line through the origin. [2]

(c) The matrix $\mathbf{N} = \begin{pmatrix} 3 & 1 \\ k & 5 \end{pmatrix}$ represents a transformation with exactly one invariant line through the origin. Find k, and the equation of that line. [4]

(d) Using this value of k, show that the origin is the only invariant point of the transformation represented by $\mathbf{N}$. [3]

(e) Still with this value of k, a triangle is transformed by $\mathbf{N}$ onto a triangle of area $48\,\text{cm}^2$. Find the area of the original triangle. [2]

Answers and fully worked solutions are available in the companion booklets.