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Additional Math with Ali Hashir

The Remainder Theorem

Worksheet

CIE O Level Additional Mathematics 4037 • Total: 52 marks

Long division of $f(x)$ by $x-a$ gives the identity $f(x) \equiv (x-a)Q(x)+R$, and this holds for *all* values of x. Choosing the single value $x=a$ makes the factor $(x-a)$ vanish, so the entire $Q(x)$ term disappears whatever $Q(x)$ happens to be, leaving $f(a)=R$. That observation is the remainder theorem, and it replaces a whole page of long division with one substitution.

Key results

$$\text{if } f(x)=(x-a)Q(x)+R, \text{ then } R=f(a); \qquad \text{if } f(x)=(ax-b)Q(x)+R, \text{ then } R=f\left(\tfrac{b}{a}\right).$$

- To find the value to substitute, **set the divisor equal to zero and solve**.
- Dividing by $x+2$ means substituting $x=-2$; dividing by $2x-3$ means substituting $x=\frac{3}{2}$.
- A quadratic divisor that factorises, such as $(x-1)(x+2)$, leaves a remainder of the form $ax+b$. Substituting each root in turn gives two equations.

Strategy for unknown coefficients. Many questions give information about the remainder and ask for missing constants.

1. Each divisor mentioned gives **one equation**. Two unknowns therefore need two divisors.
2. Substitute the root of that divisor into $f(x)$ and set the result equal to the stated remainder.
3. Solve them simultaneously, then answer what was actually asked — often a third substitution.

Watch out for.

- **Signs.** The theorem is stated for $x-a$, so $x+2$ is $x-(-2)$. Reading the sign straight off the divisor is the commonest error.
- **Non-monic divisors.** $ax-b$ requires the fraction $x=\frac{b}{a}$, not a whole number. Leave such answers as exact fractions.
- **Losing roots.** Factorise an equation such as $6p^2-2p=0$; dividing through by p throws away the solution $p=0$.

Instructions. Answer all questions using the remainder theorem rather than long division. Give exact values throughout. Marks are shown in brackets.

Q1

Find the remainder when $f(x) = x^3 - 2x^2 + 5x - 7$ is divided by $x - 2$. [3]

Q2

Find the remainder when $f(x) = 2x^3 + 3x^2 - 4x + 1$ is divided by $x + 2$. [3]

Q3

Find the remainder when $f(x) = 4x^3 - 6x^2 + 5x - 3$ is divided by $2x - 1$, giving your answer as an exact fraction. [3]

Q4

The expression $6x^2 - 2x + 3$ leaves a remainder of 3 when divided by $x - p$.
Determine the possible values of p. [4]

Q5

The expression $8x^3 + ax^2 + bx - 9$ leaves remainders of -95 and 3 when divided by $x + 2$ and $2x - 3$ respectively.
Calculate the value of a and of b. [5]

Q6

The polynomial $f(x) = x^3 + ax^2 + bx - 3$ leaves a remainder of 27 when divided by $x - 2$, and a remainder of 3 when divided by $x + 1$.
Calculate the remainder when $f(x)$ is divided by $x - 1$. [5]

Q7

The expression $x^3 + ax^2 + 7$ leaves a remainder of $2p$ when divided by $x + 1$, and a remainder of $p + 5$ when divided by $x - 2$.
Calculate the value of a and of p. [5]

Q8

When $ax^2 + bx - 6$ is divided by $x + 3$, the remainder is 9.
Find, **in terms of a only**, the remainder when $2x^3 - bx^2 + 2ax - 4$ is divided by $x - 2$. [5]

Q9

When $2x^3 - 4x^2 - 5x - 2$ is divided by $(x-1)(x+2)$, the remainder is $ax+b$. This result may be expressed as the identity

$$2x^3 - 4x^2 - 5x - 2 \equiv (x-1)(x+2)\,Q(x) + ax + b,$$

where $Q(x)$ is the quotient.

(a) State the degree of $Q(x)$, and explain why the remainder must have the form $ax+b$. [2]

(b) By substituting suitable values of x into the identity, find a and b. [3]

Q10

The polynomial $2x^4 + px^3 + qx - 4$ leaves a remainder of $36x + 32$ when it is divided by $x^2 - 2x - 3$.

Find the value of p and of q. [6]

Extra Challenge

Let $f(x) = 3x^4 - 5x^2 + 4$.

(i) Explain why the remainder when $f(x)$ is divided by $x^2 + 2$ **cannot** be found by substituting a value of x into $f(x)$. [1]

(ii) Every term of $f(x)$ involves an even power of x. By writing $u = x^2$ and dividing the resulting quadratic in u by $u + 2$, show that

$$f(x) \equiv \left(x^2 + 2\right)\left(3x^2 - 11\right) + 26,$$

and hence state the remainder when $f(x)$ is divided by $x^2 + 2$. [4]

(iii) The polynomial $g(x) = 2x^4 + px^2 + q$ leaves a remainder of 17 when divided by $x^2 + 3$, and a remainder of 7 when divided by $x^2 + 1$. Using the same substitution, find the value of p and of q. [4]

Answers available at the website.

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