Additional Math with Ali Hashir

The Factor Theorem

Worksheet

CIE O Level Additional Mathematics 4037 • Total: 62 marks

The factor theorem is the special case of the remainder theorem in which the remainder is zero. If $f(a) = 0$ then the identity $f(x) \equiv (x - a)Q(x) + f(a)$ collapses to $f(x) \equiv (x - a)Q(x)$, so $x - a$ divides $f(x)$ exactly. Read in reverse, it is the standard route into factorising a cubic or quartic: find one root by trial, then divide out to reach a quadratic you can factorise by ordinary means.

Key results

$$(x-a) \text{ is a factor of } f(x) \iff f(a) = 0, \qquad (ax-b) \text{ is a factor of } f(x) \iff f\left(\tfrac{b}{a}\right) = 0.$$

- "Exactly divisible by", "has a factor" and "leaves no remainder" all say the same thing.
- A divisor that factorises, such as $x^2 - 3x - 4 = (x - 4)(x + 1)$, gives **two** conditions — one for each root.
- If $f(x)$ has a factor $x - a$ and $g(x)$ has the same factor, then $f(a) = 0$ and $g(a) = 0$: two equations from one shared factor.

Factorising completely. Once one factor is known:

1. Divide $f(x)$ by that factor using long division. The remainder must be 0 — if it is not, check the substitution.
2. Factorise the quotient by inspection, checking its discriminant before claiming it will not factorise.

Watch out for.

- **Signs.** A factor $x + 3$ means substituting $x = -3$.
- **Non-monic factors.** For $2x + 1$, substitute $x = -\frac{1}{2}$.
- **Which letter is the variable.** A polynomial may be written in a with b as the constant; divide with respect to the stated variable.

Instructions. Answer all questions. You may use anything from the earlier worksheets: long division to find the remaining factor once one is known, and the remainder theorem wherever a non-zero remainder is involved. Marks are shown in brackets.

Q1

Show that $x-3$ is a factor of $f(x) = x^3-4x^2+x+6$, and hence factorise $f(x)$ completely. [4]

Q2

Show that $x+2$ is a factor of $f(x) = 2x^3+3x^2-8x-12$, and hence factorise $f(x)$ completely. [4]

Q3

The polynomial $f(x) = 2x^3 + x^2 + px - 4$ has a factor $x - 2$.

(i) Find p. [2]

(ii) Show that $2x + 1$ is also a factor of $f(x)$. [2]

(iii) Deduce the third factor. [2]

Q4

The polynomial $2x^2 + bx - 3$ has a factor $x - a$, where $a \neq 0$.
Express b in terms of a. [4]

Q5

The expression $x^3 + ax^2 + bx + 3$ is exactly divisible by $x + 3$, but leaves a remainder of 91 when divided by $x - 4$.
What is the remainder when it is divided by $x + 2$? [6]

Q6

The polynomials $x^3 + ax^2 - x + b$ and $x^3 + bx^2 - 5x + 3a$ have a common factor $x + 2$.
Find a and b. [5]

Q7

Show that $x^2 + 2x - 3$ is a factor of

$$(x + 2)^3 - (x + 1)^3 - 3x - 16.$$

[5]

Q8

The expression $ax^3 + bx^2 - 5x + 2a$ is exactly divisible by $x^2 - 3x - 4$.
Calculate the value of a and of b, and factorise the expression completely. [7]

Q9

For what values of p does the polynomial $(2p+1)x^2 + px + 2p^2$ have a factor

(a) $x - 1$, [2]

(b) $x + 2$? [2]

State the value of p for which it is divisible by $x + 2$ but not by $x - 1$. [2]

Q10

Given that $f(a) = a^3 - ba^2 - 4b^2a + 4b^3$, where b is a constant, show that $a - 2b$ is a factor of $f(a)$.
Find, in terms of b, the remainder when $f(a)$ is divided by $a + b$. [5]

Challenge 1

The polynomial $x^3 + ax^2 + bx + c$ is exactly divisible by $x^2 - x - 2$.
Find the value of $a + b$. [4]

Challenge 2

Given that $2x^2 + 3px - 2q$ and $x^2 + q$ have a common factor $x - a$, where p, q and a are non-zero constants, show that

$$9p^2 + 16q = 0.$$

[6]

Answers available at the website.

Instagram: **@vectorspacepk**
Website: **www.vectorspace.pk**