Additional Math with Ali Hashir

Solving Cubic Equations

Worksheet

CIE O Level Additional Mathematics 4037 • Total: 65 marks

There *is* a formula for solving a cubic — Cardano's, from the sixteenth century — and you are welcome to look it up out of interest. It is far too unwieldy to be useful, and it is not part of this syllabus, so we will not be using it. Instead the method is always the same: use the factor theorem to find *one* root by trial, divide it out, and solve the quadratic that is left.

Key results

$$f(x) = (x-\alpha)\big(px^2+qx+r\big) \quad \Longrightarrow \quad x=\alpha \ \text{ or } \ px^2+qx+r=0.$$

- If $f(\alpha)=0$ then $x-\alpha$ is a factor, and $x=\alpha$ is a root.
- A cubic has **at most three** real roots, and always **at least one**.
- The quadratic factor decides the rest: discriminant >0 gives two more roots, $=0$ gives a repeated root, and <0 gives none, leaving just one real root in total.
- Any integer root divides the constant term, so test $\pm1, \pm2, \pm3, \ldots$; for a non-monic cubic also try $\pm\frac{1}{2}$ and $\pm\frac{1}{3}$.

The method.

1. **Rearrange** to the form $f(x)=0$, expanding any brackets first.
2. **Find one root** α by trial, using the factors of the constant term.
3. **Divide** $f(x)$ by $x-\alpha$ to obtain the quadratic factor.
4. **Solve** the quadratic by factorising, or by formula if it will not factorise.

Watch out for.

- **Never cancel a common factor.** Collect everything on one side and factorise; dividing by $x+1$ destroys a root.
- **Repeated roots.** A factor $(x-2)^2$ gives one root, not two — say so.
- **Rounding.** Round only at the final step, and only when asked to.

Instructions. Answer all questions, showing the root you test and the division you carry out. Give exact answers unless told otherwise. Marks are shown in brackets.

Q1

Solve the equation
$$x^3 - 6x^2 + 11x - 6 = 0.$$
[3]

Q2

Solve the equation
$$x^3 + 2x^2 - 5x - 6 = 0.$$
[3]

Q3

Solve the equation
$$2x^3 - 3x^2 - 11x + 6 = 0.$$
[4]

Q4

Solve the equation
$$x^3 + 3x^2 = 4x + 12.$$
[4]

Q5

The curve $y = x^3$ and the line $y = 7x - 6$ intersect at three points.
Find the coordinates of these three points. [5]

Q6

Solve the equation
$$x^3 - 5x^2 + 8x - 4 = 0,$$
stating clearly how many **distinct** roots it has. [4]

Q7

Solve the equation
$$3x^3 + 5x^2 = 3x + 2,$$
giving your answers correct to two decimal places where necessary. [5]

Q8

Solve the equation

$$2x^3 - x^2 + 4x - 2 = 0,$$

and explain why it has only one real root. [5]

Q9

Factorise $f(x) = x^3 - 2x^2 - 7x - 4$ completely.

Hence solve the equation

(a) $f(x) = 0$, [2]

(b) $f(x) = (x+1)(x-4)$, [3]

(c) $f(x) = 6(x+1)$. [2]

Q10

The quartic polynomial $P(x) = x^4 + ax^3 - x^2 + bx - 12$ has factors $x - 2$ and $x + 1$.

(i) Find the value of a and of b. [4]

(ii) Hence solve the equation $P(x) = 0$. [3]

Challenge 1

The polynomial $P(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$ has $x = 3$ as a root.

(i) Find the cubic polynomial $Q(x)$ for which $P(x) = (x-3)\,Q(x)$. [3]

(ii) By finding a root of $Q(x) = 0$ by trial, solve the equation $P(x) = 0$ completely. [4]

(iii) A polynomial with integer coefficients and leading coefficient 1 has an integer root $x = k$. **Prove** that k must be a factor of the constant term, and verify that each root found in part (ii) satisfies this. [3]

Challenge 2

Given that

$$f(x) = 2x^3 + ax^2 - 7a^2x - 6a^3,$$

where a is a non-zero constant:

(i) Determine whether or not $x - a$ and $x + a$ are factors of $f(x)$. [3]

(ii) Hence factorise $f(x)$ completely. [3]

(iii) Write down, in terms of a, the roots of $f(x) = 0$. [2]

Answers available at the website.

Instagram: **@vectorspacepk**
Website: **www.vectorspace.pk**