Additional Math with Ali Hashir

Inverse Functions

Worksheet

CIE O Level Additional Mathematics 4037 • Total: 94 marks

The **inverse function** f^{-1} is the rule that undoes f. Formally,

$$\mathrm{f}^{-1}(x) = y \iff \mathrm{f}(y) = x,$$

so that $\mathrm{ff}^{-1}(x) = x$ and $\mathrm{f}^{-1}\mathrm{f}(x) = x$. Everything in this topic follows from that one sentence. In particular f^{-1} is **not** $\dfrac{1}{\mathrm{f}(x)}$ — the -1 is a label, not a power.

Reading the notation

$\mathrm{f}^{-1}(5) = 2$	means $\mathrm{f}(2) = 5$ — the same fact read backwards
$\mathrm{ff}^{-1}(x)$	$= x$, always (on the correct domain)
$\mathrm{f}^{-1}(x)$	is a *function*, not a reciprocal
$(\mathrm{fg})^{-1}$	$= \mathrm{g}^{-1}\mathrm{f}^{-1}$ — the order reverses

The first row is worth more than it looks: to find $\mathrm{f}^{-1}(5)$ you need never find f^{-1} at all — just solve $\mathrm{f}(x) = 5$.

Key results used in this worksheet

1. When an inverse exists. Only a **one-one** function has an inverse: every output must come from exactly one input. On a graph this is the *horizontal* line test. A many-to-one function such as x^2 has no inverse until its domain is restricted to one side of the vertex.

2. Finding it. Write $y = \mathrm{f}(x)$, rearrange to make x the subject, then rename the variable. If x appears twice, collect the x terms on one side and factorise.

3. Domain and range swap over.

$$\text{domain of } \mathrm{f}^{-1} = \text{range of f}, \qquad \text{range of } \mathrm{f}^{-1} = \text{domain of f}.$$

This is what tells you which square root to keep, and which value f^{-1} must exclude.

4. Reflection in $y = x$. The graph of $y = \mathrm{f}^{-1}(x)$ is the mirror image of $y = \mathrm{f}(x)$ in the line $y = x$. So for an *increasing* function, any solution of $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ must lie on $y = x$, and solving $\mathrm{f}(x) = x$ is enough.

Instructions

Answer **all** questions, showing full working — the method carries most of the marks. Give **exact** values: fractions and surds, never rounded decimals. State the excluded value(s) of x for every inverse you find, and reject any root that falls outside the domain of the original function.

Inverse Functions

Q1–Q10

Q1

Find f^{-1} in similar form for each of the following one-one functions, stating any value of x that must be excluded.

(a) $\mathrm{f} : x \mapsto 4x + 7$

(b) $\mathrm{f} : x \mapsto \dfrac{5}{x+2}, \quad x \neq -2$

(c) $\mathrm{f} : x \mapsto \dfrac{3x}{x-4}, \quad x \neq 4$

[6]

Q2

The function f is defined by $\mathrm{f} : x \mapsto 3x + 8$ for $x \in \mathbb{R}$.

(a) Find f^{-1} in similar form. [2]

(b) Find the value of $\mathrm{f}^{-1}(23)$. [1]

(c) Explain why $\mathrm{f}^{-1}\mathrm{f}(x) = x$ for every real x. [2]

Q3

The function f is defined by $\mathrm{f} : x \mapsto x^2 - 6x + 5$ for $x \geqslant 3$.

(a) Express $\mathrm{f}(x)$ in the form $(x-p)^2 + q$, and explain why f is one-one on the given domain. [3]

(b) State the range of f. [1]

(c) Find f^{-1} in similar form, and state the domain and the range of f^{-1}. [4]

Q4

Two functions are defined by

$$\mathrm{f} : x \mapsto 2x + 5 \qquad \text{and} \qquad \mathrm{g} : x \mapsto \frac{4}{x-1}, \quad x \neq 1.$$

(a) Find f^{-1} and g^{-1} in similar form. [4]

(b) Find the values of x for which $\mathrm{f}^{-1}(x) = \mathrm{g}^{-1}(x)$. [3]

Q5

Two functions are defined by

$$\mathrm{f} : x \mapsto \frac{2x}{x+3}, \quad x \neq -3 \qquad \text{and} \qquad \mathrm{g} : x \mapsto x + a,$$

where a is a constant.

(a) Find f^{-1} in similar form. [3]

(b) Given that $\mathrm{gf}^{-1}(4) = 1$, find the value of a. [2]

(c) State the value of x for which f^{-1} is not defined, and explain what this tells you about the range of f. [3]

Q6

The function f is defined by

$$\mathrm{f} : x \mapsto 3x - \frac{2}{x}, \qquad x > 0.$$

(a) Find the value of $\mathrm{f}^{-1}(5)$. [3]

(b) Find the value of $\mathrm{f}^{-1}(-1)$. [2]

(c) In each part above a second root appeared and had to be discarded. Explain, in terms of the domain of f, why it must be rejected. [2]

Q7

The function f is defined by

$$\mathrm{f} : x \mapsto \frac{b}{x+2}, \qquad x \neq -2,$$

where b is a non-zero constant.

(a) Find f^{-1} in similar form. [3]

(b) Given that $\mathrm{f}(3) + \mathrm{f}^{-1}(1) = 4$, find the value of b. [3]

Q8

Given the functions $\mathrm{f} : x \mapsto 2x - 3$, $x \in \mathbb{R}$, and $\mathrm{g} : x \mapsto \dfrac{1}{x-1}$ (for $x \neq 1$), find in similar form

(a) fg and $(\mathrm{fg})^{-1}$, [4]

(b) f^{-1}, g^{-1} and $\mathrm{g}^{-1}\mathrm{f}^{-1}$. [4]

Is $(\mathrm{fg})^{-1} = \mathrm{g}^{-1}\mathrm{f}^{-1}$? [1]

Q9

The function f is defined by

$$\mathrm{f} : x \mapsto \frac{3x+a}{x-2}, \qquad x \neq 2,$$

where a is a constant.

(a) Find f^{-1} in similar form, in terms of a. [3]

(b) Given that $\mathrm{f}^{-1}(5) = 8$, find the value of a. [2]

(c) Using this value of a, solve the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [4]

Q10

The functions f and g are defined by

$$\mathrm{f} : x \mapsto 3x - 4, \quad x \in \mathbb{R} \qquad \text{and} \qquad \mathrm{g} : x \mapsto x^3 + ax^2 + bx + 6,$$

where a and b are constants.

(a) Find f^{-1} in similar form. [2]

(b) It is given that $\mathrm{gf}^{-1}(2) = 0$, and that $\mathrm{g}(x)$ leaves a remainder of 24 when divided by $x - 3$. Find the value of a and of b. [5]

(c) Hence solve the equation $\mathrm{g}(x) = 0$, giving your answers in exact form. [3]

Challenge questions

Challenge 1

The function f is defined by

$$\mathrm{f} : x \mapsto \frac{ax+b}{cx-a}, \qquad x \neq \frac{a}{c},$$

where a, b and c are constants with $c \neq 0$ and $a^2 + bc \neq 0$.

(a) Show that f is *self-inverse*, that is, $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. [4]

(b) Hence write down expressions for $\mathrm{f}^2(x)$ and $\mathrm{f}^5(x)$. [2]

(c) Verify that $\mathrm{f} : x \mapsto \dfrac{2x+5}{x-2}$, $x \neq 2$, is of this form, and find the values of x for which $\mathrm{f}(x) = x$. [3]

Challenge 2

(a) By simplifying $\mathrm{fg}\big(\mathrm{g}^{-1}\mathrm{f}^{-1}(x)\big)$, prove that $(\mathrm{fg})^{-1} = \mathrm{g}^{-1}\mathrm{f}^{-1}$ for any two one-one functions f and g. [4]

(b) Explain in one sentence why the order reverses. [2]

(c) The function h is defined by $\mathrm{h} : x \mapsto 5(2x-1)^3 + 4$ for $x \in \mathbb{R}$. By regarding h as a chain of simple steps applied to x, write down $\mathrm{h}^{-1}(x)$ by undoing those steps in reverse order. [4]

Answers and full worked solutions available at the website.

Instagram: **@vectorspacepk**
Website: **www.vectorspace.pk**