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Additional Math with Ali Hashir

Basic Functions

Worksheet

CIE O Level Additional Mathematics 4037 • Total: 123 marks

A **function** is a rule that gives each input *exactly one* output, and every "is this a function?" question is settled by that sentence alone. Two inputs *may* share an image (many-to-one is fine); one input may *not* have two (one-to-many is not a function). On a graph this is the **vertical line test** — a *horizontal* line meeting the curve twice is no objection.

Reading the notation

$\mathrm{f} : x \mapsto 3x - 1$	the *rule*: "f sends x to $3x - 1$"
$\mathrm{f}(2) = 5$	"the *image* of 2 is 5", i.e. 2 is sent to 5
domain	the set of inputs f is allowed to act on
range (image set)	the set of *outputs* actually produced

Watch the direction of travel: "the image of 3" means substitute $x = 3$, but "the number *whose* image is 3" means solve $\mathrm{f}(x) = 3$ — an equation, which may have two solutions, one, or none at all.

Key results used in this worksheet

1. Excluded values. $\dfrac{1}{x-k}$ excludes $x = k$; $\sqrt{g(x)}$ needs $g(x) \geqslant 0$; $\dfrac{1}{\sqrt{g(x)}}$ needs $g(x) > 0$ (the zero is lost too).

2. Range of a quadratic. Complete the square: $\mathrm{f}(x) = a(x-b)^2 + c$ has its vertex at $x = b$, $\mathrm{f}(b) = c$ — a *minimum* if $a > 0$, a *maximum* if $a < 0$.

3. Range on a restricted domain — the common trap. On $m \leqslant x \leqslant n$ the extreme outputs come from *three* candidates: $\mathrm{f}(m)$, $\mathrm{f}(n)$ and the vertex — but the vertex counts **only if it lies inside the domain**.

4. Rational functions. Rewriting $\dfrac{px+q}{x-r}$ as $A + \dfrac{B}{x-r}$ exposes the behaviour: $\dfrac{B}{x-r}$ gets as close to 0 as you like but never reaches it, so A is never an output.

Instructions

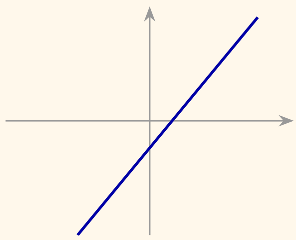
Answer **all** questions, showing full working — the method carries most of the marks. Give **exact** values: fractions and surds, never rounded decimals. State the excluded value(s) of x whenever a function is written as a fraction or a root, and sketch the curve before answering any question about a range.

Section A — Functions, mappings and images Q1–Q10

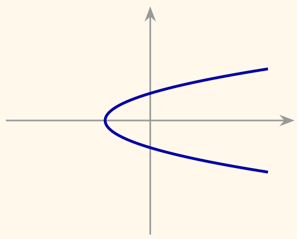
Q1

Each graph below shows a relation between x and y. For each one, state whether the relation is a function of x, and give a reason for your answer. [5]

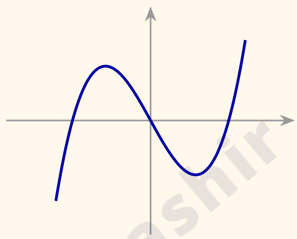
(a)



(b)



(c)



For part (c), be careful: the graph is not a straight line, but that on its own decides nothing.

Q2

The function f is defined by $\mathrm{f} : x \mapsto 5 - 3x$ for the domain $D = \{-2, 0, 1, 4\}$.

(a) Write down the image set of f. [3]

(b) The number p is added to the domain, and it is found that $\mathrm{f}(p) = -13$. Find p. [2]

Q3

A function g is defined by

$$\mathrm{g} : x \mapsto 4 + \frac{2}{x-2}, \qquad x \neq 2.$$

(a) Find the images of 0, 3 and 6. [3]

(b) Find the number whose image is 5. [3]

Q4

A function f is defined by $\mathrm{f} : x \mapsto ax + b$, where a and b are constants. The images of 2 and -3 are 1 and 16 respectively.

(a) Find the value of a and of b. [3]

(b) For the domain $X = \{-1, 0, 4\}$, find the range of f. [2]

(c) Find the value of x that is its own image under f. [2]

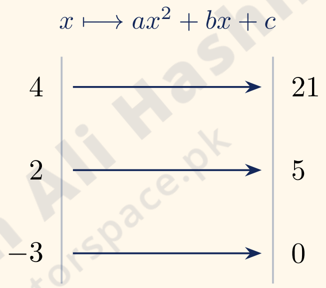
Q5

The arrow diagram shows part of the function $\mathrm{f} : x \mapsto ax^2 + bx + c$, where a, b and c are constants.

(a) Find the value of a, of b and of c. [4]

(b) Find the positive number whose image is 12. [2]

(c) Find the set of *all* numbers whose image under f is 12. [2]



Q6

The mapping f is defined by

$$\mathrm{f} : x \mapsto \frac{6}{ax+b}, \qquad x \neq -\frac{b}{a}, \quad a \neq 0,$$

where a and b are constants. It is given that f maps 1 to 2 and maps -2 to -2.

(a) Find the value of a and of b. [4]

(b) Find the element whose image is 4. [2]

(c) Explain why no element of the domain has image 0. [2]

Q7

A function h is defined by $\mathrm{h} : x \mapsto 5 - 2x$.

(a) Find, in terms of a and in its simplest form, an expression for

$$\text{(i) } \mathrm{h}(2a+3), \qquad \text{(ii) } \mathrm{h}(a^2 - a), \qquad \text{(iii) } \mathrm{h}\left(\frac{1}{a}\right), \qquad \text{(iv) } \mathrm{h}\left(\frac{a}{a-1}\right).$$

[6]

(b) Find the values of a for which $\mathrm{h}(a^2) = \mathrm{h}(a)$. [2]

Q8

Two functions f and g are defined by

$$\mathrm{f} : x \mapsto \frac{2}{x-3}, \quad x \neq 3, \qquad \text{and} \qquad \mathrm{g} : x \mapsto \frac{x}{x+2}, \quad x \neq -2.$$

Find the values of x for which $\mathrm{f}(x) = 3\,\mathrm{g}(x)$. [6]

Q9

The function f is defined by

$$\mathrm{f} : x \mapsto \frac{3x-1}{x+1}, \quad x \neq -1.$$

(a) Show that there is exactly one value of x which is equal to its own image, and state that value. [4]

(b) Find the values of x for which $\mathrm{f}(x) = \dfrac{1}{x}$. Comment on how one of your answers relates to part (a). [3]

Q10

Two functions are defined by

$$f : x \mapsto ax + 2 \qquad \text{and} \qquad g : x \mapsto \frac{6b}{x+1}, \quad x \neq -1,$$

where a and b are non-zero constants. Given that

$$f(a) = g(b) \qquad \text{and} \qquad f\left(\frac{1}{a}\right) = g\left(\frac{1}{b}\right),$$

find the possible values of a and of b. [8]

Section B — Domain and range

Q11–Q15

Q11

State the largest possible domain of real values of x for each of the following functions.

(a) $\mathrm{f} : x \mapsto \dfrac{1}{2x-7}$

(b) $\mathrm{g} : x \mapsto \sqrt{x^2 - x - 6}$

(c) $\mathrm{h} : x \mapsto \dfrac{3}{\sqrt{5-x}}$

(d) $\mathrm{p} : x \mapsto \dfrac{x+1}{x^2-9}$

[7]

Q12

The function f is defined by $\mathrm{f} : x \mapsto x^2 - 6x + 5$ for the domain $0 \leqslant x \leqslant 5$.

(a) Express $\mathrm{f}(x)$ in the form $(x-p)^2 + q$. [2]

(b) Find the range of f. [3]

(c) State a *different* domain of the form $m \leqslant x \leqslant n$ for which f has exactly the same range. [2]

Q13

(a) For the function $\mathrm{f} : x \mapsto 9 - 2x^2$ with domain $x \geqslant 0$, find the domain of x corresponding to the range $1 \leqslant \mathrm{f}(x) \leqslant 7$. [3]

(b) The function g is defined by $\mathrm{g} : x \mapsto (x-2)^2$. Find the range of g corresponding to the domain

(i) $3 \leqslant x \leqslant 6$, (ii) $-1 \leqslant x \leqslant 3$.

[4]

In part (b), the two answers are not obtained by the same shortcut. Sketch the curve before writing anything down.

Q14

The function f is defined by $\mathrm{f} : x \mapsto 2x^2 - 8x + 11$.

(a) Express $\mathrm{f}(x)$ in the form $a(x-b)^2 + c$. [3]

(b) Write down the range of f for the domain $x \geqslant 0$. [2]

(c) Given instead that the domain is $0 \leqslant x \leqslant k$ and that the corresponding range is $3 \leqslant \mathrm{f}(x) \leqslant 19$, find the exact value of k. [3]

Q15

The function f is defined by

$$\mathrm{f} : x \mapsto \frac{2x+1}{x-3}, \qquad x \neq 3.$$

(a) Show that $\mathrm{f}(x)$ may be written as $2 + \dfrac{7}{x-3}$. [2]

(b) Hence explain why there is no value of x for which $\mathrm{f}(x) = 2$. [2]

(c) Find the range of f for the domain $4 \leqslant x \leqslant 10$. [2]

(d) Find the set of values of x for which $\mathrm{f}(x) \geqslant 5$. [3]

Challenge questions

Challenge 1

A function f is defined for *all* real values of x, and satisfies

$$\mathrm{f}(x) + 2\,\mathrm{f}(-x) = 3x + 9 \qquad \text{for every real number } x.$$

(a) Find an expression for $\mathrm{f}(x)$. [5]

(b) Hence find the value of x which is equal to its own image. [2]

The statement is true for every *x, so you are free to replace x by anything you like — and then you have two equations.*

Challenge 2

The function f is defined by $\mathrm{f} : x \mapsto x^2 - 2kx + k^2 + 3k$ for the domain $x \geqslant 0$, where k is a constant.

(a) Show that $\mathrm{f}(x) = (x - k)^2 + 3k$. [2]

(b) Given that $k > 0$, write down the range of f. [2]

(c) Given instead that $k < 0$, find the range of f in terms of k. [3]

(d) Find the set of values of k for which the range of f contains no negative numbers. [3]

Part (c) is not part (b) with a sign changed. Ask yourself whether the vertex of the parabola is actually inside the domain.

Answers and full worked solutions available at the website.

Instagram: **@vectorspacepk**
Website: **www.vectorspace.pk**