Additional Math with Ali Hashir

Absolute Value (Modulus) Functions

Worksheet

CIE O Level Additional Mathematics 4037 • Total: 150 marks

The **absolute value** (or **modulus**) of a number strips away its sign:

$$|x| = \begin{cases} x & \text{if } x \geqslant 0, \\ -x & \text{if } x < 0, \end{cases} \qquad \text{so } |x| \text{ is the } \textit{distance} \text{ of } x \text{ from } 0.$$

Two consequences run through the whole topic: $|x|$ is **never negative**, and one value of $|x|$ comes from **two** possible values of x. Nearly every question below applies one of those two facts.

Reading the notation

$\lvert -7\rvert = 7$	the sign is discarded, not the number
$\lvert A\rvert = k \quad (k > 0)$	has *two* solutions: $A = k$ or $A = -k$
$\lvert A\rvert = k \quad (k < 0)$	has *no* solutions
$y = \lvert f(x)\rvert$	$y = f(x)$ with everything below the x-axis reflected *up*

Note that $|f(x)|$ means "work out $f(x)$ first, then remove the sign" — so $|x^2 - 5x|$ is not $x^2 + 5x$.

Key results used in this worksheet

1. Solving $|A| = k$. Split into the two cases $A = k$ and $A = -k$ and solve each. If $k < 0$ there is nothing to do: the equation has no solutions.

2. Solving $|A| = B$, where B contains x. Split as before into $A = B$ and $A = -B$, **but then check every root**: it is valid only if it makes $B \geqslant 0$, since the left side can never be negative. Discarding false roots is where the marks are.

3. Squaring. $|A| = |B|$ is exactly equivalent to $A^2 = B^2$, so squaring is safe when *both* sides are moduli. When only one is, squaring can manufacture false roots.

4. Inequalities. For $k > 0$,

$$|A| \leqslant k \iff -k \leqslant A \leqslant k \quad \text{(inside)}, \qquad |A| \geqslant k \iff A \geqslant k \quad \text{or} \quad A \leqslant -k \quad \text{(outside)}.$$

A sketch settles which of the two shapes you need.

Instructions

Answer **all** questions, showing full working, and give **exact** values. Set out modulus equations as two clearly labelled cases, and say explicitly whenever a root is rejected and why. Sketches need not be to scale, but must show the vertex, the axis intercepts and, where a domain is given, the endpoints.

Absolute Value Functions

Q1–Q10

Q1

A function f is defined by $f : x \mapsto |4x - 5|$ for $x \in \mathbb{R}$.

(a) Find $f(-2)$, $f(1)$ and $f(3)$. [3]

(b) Find the values of x for which $f(x) = 7$. [2]

(c) Explain why the equation $f(x) = -2$ has no solutions. [1]

(d) Sketch the graph of $y = f(x)$, showing the coordinates of the vertex and of both axis intercepts. [3]

(e) State the range of f. [1]

Q2

(a) Solve $|3x - 4| = 11$. [2]

(b) Solve $|x - 6| = |2x + 1|$. [3]

(c) Solve $|5 - 2x| = 0$, and explain why this equation has only *one* solution when the equations in parts (a) and (b) each have two. [2]

(d) Explain why $|x - 1| = |1 - x|$ for every value of x. [2]

Q3

Solve each of the following equations. In every case check your roots and state clearly any root that must be rejected, together with the reason.

(a) $|2x + 1| = x + 5$ [3]

(b) $|x - 3| = 2x - 8$ [4]

(c) $|3x - 2| = x - 4$ [3]

(d) Sketch, on a single diagram, the graphs of $y = |2x + 1|$ and $y = x + 5$, and explain how the diagram confirms your answer to part (a). [3]

Q4

(a) Show that the equation $|x+3| = x^2 + 1$ splits into two quadratic equations, and solve it completely. [5]

(b) Sketch, on a single diagram, the graphs of $y = |x+3|$ and $y = x^2 + 1$, marking the points of intersection. [3]

(c) Hence, or otherwise, find the set of values of x for which $|x+3| > x^2 + 1$. [3]

Q5

The function f is defined by $\mathrm{f} : x \mapsto |2x - 6|$ for the domain $0 \leqslant x \leqslant 5$.

(a) Sketch the graph of $y = \mathrm{f}(x)$, showing the vertex and both endpoints. [3]

(b) State the range of f. [2]

(c) State, with a reason, whether f is a one-one function. [2]

(d) Find the largest value of k for which f *is* one-one on the restricted domain $0 \leqslant x \leqslant k$. [2]

(e) Solve the equation $\mathrm{f}(x) = 4$. [2]

Q6

The function f is defined by $\mathrm{f} : x \mapsto |3x + 6| - 4$ for $x \in \mathbb{R}$.

(a) Sketch the graph of $y = \mathrm{f}(x)$, showing the coordinates of the vertex and of all three axis intercepts. [3]

(b) State the range of f. [1]

(c) Find the range of values of x for which $\mathrm{f}(x) \leqslant 5$. [3]

(d) Find the range of values of x for which $\mathrm{f}(x) > 2$. [3]

(e) State the number of solutions of the equation $\mathrm{f}(x) = k$ when

$$\text{(i) } k = -6, \qquad \text{(ii) } k = -4, \qquad \text{(iii) } k = -1.$$

[3]

Q7

This question is about solving simultaneous equations in which one equation involves a modulus.

(a) Sketch, on a single diagram, the graphs of $y = |x-2|$ and $y = 3x-4$, and hence solve the simultaneous equations $y = |x-2|$ and $y = 3x-4$. [5]

(b) Explain why the line $y = x-4$ does not meet the graph of $y = |x-2|$ at all. [3]

(c) Solve the simultaneous equations $y = |2x+1|$ and $y = x+7$. [4]

(d) Solve the simultaneous equations $y = |x|$ and $y = x^2-2$, and sketch the two graphs on a single diagram to confirm the number of solutions. [4]

Q8

The function f is defined by $\mathrm{f} : x \mapsto \left|x^2 - 5x\right|$ for $x \in \mathbb{R}$.

(a) Find all the values of x for which $\mathrm{f}(x) = 6$. [4]

(b) Find all the values of x for which $\mathrm{f}(x) = x$. [4]

(c) Sketch the graph of $y = \mathrm{f}(x)$, showing the x-intercepts and the coordinates of the highest point of the reflected arch. [3]

(d) Hence state the number of solutions of $\mathrm{f}(x) = k$ when

$$\text{(i) } 0 < k < \tfrac{25}{4}, \qquad \text{(ii) } k = \tfrac{25}{4}, \qquad \text{(iii) } k > \tfrac{25}{4}.$$

[3]

Q9

A function f is defined by $\mathrm{f} : x \mapsto |x-a| + b$ for $x \in \mathbb{R}$, where a and b are constants. It is given that $\mathrm{f}(1) = 7$ and $\mathrm{f}(9) = 7$.

(a) Find the value of a and of b. [4]

(b) State the least value of f, and the value of x at which it occurs. [2]

(c) Sketch the graph of $y = \mathrm{f}(x)$, showing the vertex and the points $(1, 7)$ and $(9, 7)$. [2]

(d) Solve the equation $\mathrm{f}(x) = 10$. [3]

(e) State the set of values of k for which the equation $\mathrm{f}(x) = k$ has exactly two solutions. [2]

Q10

The function f is defined by $f : x \mapsto |x^2 - 4|$ for the domain $-3 \leqslant x \leqslant 3$.

(a) Sketch the graph of $y = f(x)$, showing the x-intercepts, the y-intercept and both endpoints. [3]

(b) State the range of f. [2]

(c) Using your sketch, state how many solutions the equation $f(x) = k$ has when

(i) $k = 0$, (ii) $k = 3$, (iii) $k = 5$,

giving a brief reason in each case. [4]

(d) Solve the equation $f(x) = 3$, giving your answers in exact form. [3]

(e) Find the set of values of x for which $f(x) < 3$. [3]

Challenge questions

Challenge 1

(a) By splitting the real line into three regions, solve the equation

$$|x-1|+|x+2|=5.$$

[6]

(b) Show that $|x-1|+|x+2|$ takes the same value for every x with $-2 \leqslant x \leqslant 1$, and interpret this fact in terms of distances along the number line. [3]

(c) Hence state, with reasons, the values of k for which the equation $|x-1|+|x+2|=k$ has (i) no solutions, (ii) infinitely many solutions, (iii) exactly two solutions. [3]

Challenge 2

(a) By squaring both sides, solve the equation $|2x-1|=|x+4|$. [4]

(b) The same method is now applied to $|x+2|=x-4$. Show that squaring both sides produces $x=1$, and demonstrate that this is *not* a solution of the original equation. [3]

(c) Explain why squaring is always safe in part (a) but not in part (b), and state the check that must accompany the method whenever only one side is a modulus. [2]

(d) Solve the equation $|3x-1|=|x+5|$, and hence find the set of values of x for which $|3x-1|<|x+5|$. [4]

Answers and full worked solutions available at the website.

Instagram: **@vectorspacepk**
Website: **www.vectorspace.pk**