Additional Math with Ali Hashir

The Binomial Theorem

Worksheet 2: General Expansions of $(a+b)^n$

O Level Additional Mathematics 4037 • Binomial Expansions

Summary. For a positive integer n, the binomial theorem states that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n}b^n,$$

where

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}.$$

Unlike Worksheet 1, the leading term a is no longer 1, so every term carries a power of a as well as a power of b: the general term (the $(r+1)$th term) is

$$T_{r+1} = \binom{n}{r}\,a^{n-r}\,b^r.$$

Key facts.

- **Powers balance.** In every term of $(a+b)^n$ the powers of a and b add up to n. Writing the general term first and tidying the powers of x *before* substituting values of r saves enormous amounts of work.
- **Negative and fractional entries.** Terms such as $-2x$, $\dfrac{x}{2}$ or $\dfrac{2}{x}$ are handled by the same formula – keep the sign and the fraction inside the bracket when raising to a power.
- **Terms with $1/x$.** When the bracket contains both x and $\dfrac{1}{x}$ (or x^2, x^3, $\ldots$), the power of x in the general term is a linear expression in r. Set it equal to the power you want – including 0 for the *constant term* – and solve for r.
- **Estimation.** Substituting a small value of x into the first few terms of an expansion gives accurate numerical approximations.

Instructions. Answer all questions, showing full working. Final coefficients should be evaluated as exact numbers. Questions are arranged in increasing order of difficulty, finishing with two challenge questions. Calculators may be used unless a question says otherwise.

Q1

(a) Write out the full expansion of $(2+x)^4$, simplifying each term. [2]

(b) Find, in ascending powers of x, the first 4 terms in the expansion of $(2-3x)^6$. [3]

Q2

Find, in ascending powers of x, the first 4 terms in the expansion of

$$\left(4+\frac{x}{2}\right)^5.$$

[3]

Q3

Obtain, in ascending powers of x, the first 4 terms in the expansion of

$$\left(3-\frac{x}{3}\right)^6.$$

Hence find the value of $(2.99)^6$ correct to 4 decimal places. [5]

Q4

Find the indicated term in each of the following expansions.

(a) $(3+x)^8$, 6th term [2]

(b) $(2x-1)^9$, 4th term [2]

(c) $(y-3x)^8$, 5th term [2]

(d) $\left(2x-\frac{1}{x}\right)^{10}$, middle term [2]

Q5

Find the non-zero values of a and b for which

$$(3x-a)^3 = 27x^3 - bx^2 + \frac{4}{3}bx - a^3.$$

[4]

Q6

In the expansion of

$$\left(x^2 + \frac{2}{x}\right)^9,$$

find the term in x^3 and the constant term. [5]

Q7

In the expansion of $\left(x^3 - \frac{3}{x}\right)^8$, find

(a) the term in x^8, [2]

(b) the coefficient of $\frac{1}{x^4}$, [2]

(c) the constant term. [2]

Q8

Expand $(3-2x)^5$ in ascending powers of x up to the term in x^3. Given that the coefficient of x^2 in the expansion of

$$(1+ax+2x^2)(3-2x)^5$$

is -54, find the value of the constant a, and hence find the coefficient of x^3. [6]

Q9

In the expansion of $(1 + 2x)(a - bx)^{10}$, where $ab \neq 0$, the coefficient of x^6 is zero. Find, in its simplest form, the value of the ratio $\frac{a}{b}$. [5]

Q10

Find the expansions of $(2 + x)^6$ and $(3 - x)^6$ in ascending powers of x as far as the terms in x^2. Hence expand

$$(6 + x - x^2)^6$$

up to the term in x^2. [6]

Challenge 1

In the expansion of $(2+x)^n$, where n is a positive integer, the coefficients of x^4 and x^5 are equal.

(a) Find the value of n. [4]

(b) For this value of n, show that the ratio of the coefficient of x^6 to the coefficient of x^4 is $3:4$. [3]

Challenge 2

(a) Explain why, in the sum $(2+x)^5+(2-x)^5$, the terms in odd powers of x cancel, and show that

$$(2+x)^5+(2-x)^5 = 20x^4+160x^2+64.$$

[4]

(b) Hence solve the equation

$$(2+x)^5+(2-x)^5 = 464,$$

giving your answers in exact form. [4]

Answers and full worked solutions available separately.