Additional Math with Ali Hashir

The Binomial Theorem

Worksheet 1: Expansions of the Form $(1+bx)^n$

O Level Additional Mathematics 4037 • Binomial Expansions

Summary. The binomial theorem gives the expansion of a two-term power without repeated multiplication. For a positive integer n,

$$(1+b)^n = \binom{n}{0} + \binom{n}{1}b + \binom{n}{2}b^2 + \cdots + \binom{n}{r}b^r + \cdots + \binom{n}{n}b^n,$$

where the binomial coefficients are

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}.$$

In this worksheet every bracket has leading term 1, so each expansion has the clean shape

$$(1+bx)^n = 1 + \binom{n}{1}(bx) + \binom{n}{2}(bx)^2 + \binom{n}{3}(bx)^3 + \cdots,$$

with general term $\binom{n}{r}(bx)^r$.

Key facts.

- **Symmetry.** $\binom{n}{r} = \binom{n}{n-r}$, so coefficients read the same from both ends of a full expansion.
- **Counting terms.** $(1+bx)^n$ has $n+1$ terms; the $(r+1)$th term is $\binom{n}{r}(bx)^r$. For even n the middle term is the $\left(\frac{n}{2}+1\right)$th.
- **Signs.** If $b<0$ the signs alternate: $+,-,+,-,\ldots$
- **Powers inside powers.** In $(1+bx^2)^n$ the general term is $\binom{n}{r}b^r x^{2r}$ – only even powers of x can appear.
- **Estimation.** Substituting a small value of x into the first few terms gives quick numerical approximations, because higher powers of a small number are negligible.

Instructions. Answer all questions, showing full working. Unsimplified binomial coefficients such as $\binom{12}{6}$ may appear in your working, but final coefficients should be evaluated. Questions are arranged in increasing order of difficulty, finishing with two challenge questions. Calculators may be used unless a question says otherwise.

Q1

(a) Write out the full expansion of $(1+3x)^4$, simplifying each term. [2]

(b) Find, in ascending powers of x, the first 4 terms in the expansion of $(1-2x)^7$. [3]

Q2

Find, in ascending powers of x, the first 3 terms in the expansion of

$$\left(1+\frac{x}{2}\right)^{10}.$$

By choosing a suitable value of x, use your expansion to estimate $(1.005)^{10}$ correct to 4 decimal places. [4]

Q3

In each of the following expansions, find the indicated term.

(a) $(1+4x)^8$, 6th term [2]

(b) $(1-2x)^{12}$, middle term [2]

(c) $(1+3x^2)^7$, term in x^8 [2]

Q4

In the expansion of

$$\left(1-\frac{x^2}{2}\right)^8,$$

find the coefficient of x^6 and the coefficient of x^{10}. Explain briefly why no term in x^7 appears in the expansion. [5]

Q5

In the expansion of $(1+3x)^{10}$, the coefficient of x^4 is k times the coefficient of x^2. Evaluate k. [4]

Q6

Expand $(1+2x)^{18}+(1-2x)^{18}$ in ascending powers of x up to the term in x^4, explaining why only even powers of x occur. Use your result to evaluate

$$(1.02)^{18}+(0.98)^{18}$$

correct to 3 significant figures. [5]

Q7

(a) Find the ratio of the coefficient of the 5th term to the coefficient of the 3rd term in the expansion of $(1+2x)^{14}$. [3]

(b) The coefficient of x^2 in the expansion of $(1-ax)^{20}$, where $a>0$, is 760. Find the coefficient of x^3. [3]

Q8

When $(1+ax)^n$ is expanded in ascending powers of x, where n is a positive integer and a is a constant, the first three terms are

$$1+15x+90x^2.$$

Find the value of n and the value of a, and hence find the coefficient of x^3. [5]

Q9

When $(1+2x)(1-ax)^6$ is expanded as far as the term in x^2, the result is $1+bx^2$; that is, the expansion contains no term in x. Find the value of a and of b. [5]

Q10

If x is sufficiently small such that the terms in x^3 and higher powers of x may be neglected, show that

$$(1+x)^5(1-4x)^4 \approx 1-11x+26x^2.$$

[4]

Hence or otherwise, expand the following up to the term in x^2.

(a) $(1-x)^5(1+4x)^4$ [2]

(b) $(1+x^2)^5(1-2x)^4(1+2x)^4$ [3]

Challenge 1

Without expanding any of the powers individually, show that

$$(1-2x)^5 + 5(1-2x)^4(2x) + 10(1-2x)^3(2x)^2 + 10(1-2x)^2(2x)^3 + 5(1-2x)(2x)^4 + (2x)^5$$

is equal to 1 for every value of x.
Hint: compare the expression with the expansion of $(a+b)^5$ for a clever choice of a and b.[4]

Challenge 2

By writing 9 as $1+8$ and using the binomial theorem, show that

$$9^n + 7$$

is divisible by 8 for every positive integer n.
Deduce the remainder when $9^{2026} + 2026$ is divided by 8. [5]

Answers and full worked solutions available separately.

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