Further Math with Ali Hashir

Probability Generating Functions

Topical Questions (CIE 9231)

Compiled from CIE 9231 Further Mathematics past papers (2015–2025), sorted by topic. Each question is numbered for this booklet, with its original paper code and session shown beneath.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Q1

9231/21, 9231/22 • May/June 2017 • Question 6

A fair die is thrown repeatedly until a 6 is obtained.

(i) Find the probability that obtaining a 6 takes no more than four throws. [2]

(ii) Find the least integer N such that the probability of obtaining a 6 before the Nth throw is more than 0.95. [3]

Q2

9231/21, 9231/22 • May/June 2018 • Question 9

At a ski resort, the probability of snow on any particular day is constant and equal to p. The skiing season begins on 1 November. The random variable X denotes the day of the skiing season on which the first snowfall occurs. (For example, if the first snowfall is on 5 November, then $X = 5$.) The variance of X is $\frac{4}{9}$.

(i) Show that $4p^2 + 9p - 9 = 0$ and hence find the value of p. [4]

(ii) Find the probability that the first snowfall will be on 3 November. [1]

(iii) Find the probability that the first snowfall will not be before 4 November. [2]

(iv) Find the least integer N so that the probability of the first snowfall being on or before the Nth day of November is more than 0.999. [4]

Q3

9231/23 • May/June 2018 • Question 7

The probability that a driver passes an advanced driving test has a fixed value p for each attempt. A driver keeps taking the test until he passes. The random variable X denotes the number of attempts required for the driver to pass. The variance of X is 3.75.

(i) Show that $15p^2 + 4p - 4 = 0$ and hence find the value of p. [4]

(ii) Find $P(X = 5)$. [1]

(iii) Find $P(3 \leq X \leq 7)$. [2]

Q4

9231/21, 9231/22 • May/June 2019 • Question 6

A fair six-sided die is thrown until a 3 or a 4 is obtained. The number of throws taken is denoted by the random variable X.

(i) State the mean value of X. [1]

(ii) Find the probability that obtaining a 3 or a 4 takes exactly 6 throws. [1]

(iii) Find the probability that obtaining a 3 or a 4 takes more than 4 throws. [2]

(iv) Find the greatest integer n such that the probability of obtaining a 3 or a 4 in fewer than n throws is less than 0.95. [3]

Q5

9231/23 • May/June 2019 • Question 7

A pair of fair coins is thrown repeatedly until a pair of tails is obtained. The number of throws taken is denoted by the random variable X.

(i) State the expected value of X. [1]

(ii) Find the probability that exactly 3 throws are required to obtain a pair of tails. [2]

(iii) Find the probability that fewer than 4 throws are required to obtain a pair of tails. [2]

(iv) Find the least integer N such that the probability of obtaining a pair of tails in fewer than N throws is more than 0.95. [3]

Q6

9231/41, 9231/42 • May/June 2020 • Question 6

A bag contains 4 red balls and 6 blue balls. Rassa selects two balls at random, without replacement, from the bag. The number of red balls selected by Rassa is denoted by X.

(a) Find the probability generating function, $\mathrm{G}_X(t)$, of X. [2]

Rassa also tosses two coins. One coin is biased so that the probability of a head is $\frac{2}{3}$. The other coin is biased so that the probability of a head is p. The probability generating function of Y, the number of heads obtained by Rassa, is $\mathrm{G}_Y(t)$. The coefficient of t in $\mathrm{G}_Y(t)$ is $\frac{7}{12}$.

(b) Find $\mathrm{G}_Y(t)$. [3]

The random variable Z is the sum of the number of red balls selected and the number of heads obtained by Rassa.

(c) Find the probability generating function of Z, expressing your answer as a polynomial. [3]

(d) Use the probability generating function of Z to find $\mathrm{E}(Z)$. [2]

Q7

9231/43 • May/June 2020 • Question 4

The discrete random variable X has probability generating function $\mathrm{G}_X(t)$ given by

$$\mathrm{G}_X(t) = 0.2t + 0.5t^2 + 0.3t^3.$$

The random variable Y is the sum of two independent observations of X.

(a) Find the probability generating function of Y, giving your answer as an expanded polynomial in t. [3]

(b) Use the probability generating function of Y to find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. [5]

Q8

9231/41, 9231/43 • October/November 2020 • Question 5

Keira has two unbiased coins. She tosses both coins. The number of heads obtained by Keira is denoted by X.

(a) Find the probability generating function $\mathrm{G}_X(t)$ of X. [1]

Hassan has three coins, two of which are biased so that the probability of obtaining a head when the coin is tossed is $\frac{1}{3}$. The corresponding probability for the third coin is $\frac{1}{4}$. The number of heads obtained by Hassan when he tosses these three coins is denoted by Y.

(b) Find the probability generating function $\mathrm{G}_Y(t)$ of Y. [3]

The random variable Z is the total number of heads obtained by Keira and Hassan.

(c) Find the probability generating function of Z, expressing your answer as a polynomial. [3]

(d) Use the probability generating function of Z to find $\mathrm{E}(Z)$. [2]

(e) Use the probability generating function of Z to find the most probable value of Z. [1]

Q9

9231/42 • October/November 2020 • Question 5

The random variable X has the binomial distribution $B(n, p)$.

(a) Write down an expression for $P(X = r)$ and hence show that the probability generating function of X is $(q + pt)^n$, where $q = 1 - p$. [3]

(b) Use the probability generating function of X to prove that $E(X) = np$ and $Var(X) = np(1 - p)$. [5]

Q10

9231/41, 9231/42 • May/June 2021 • Question 6

Tanji has a bag containing 4 red balls and 2 blue balls. He selects 3 balls at random from the bag, without replacement. The number of red balls selected by Tanji is denoted by X.

(a) Find the probability generating function $\mathrm{G}_X(t)$ of X. [2]

Tanji also has two coins, each biased so that the probability of obtaining a head when it is thrown is $\frac{1}{4}$. He throws the two coins at the same time. The number of heads obtained is denoted by Y.

(b) Find the probability generating function $\mathrm{G}_Y(t)$ of Y. [2]

The random variable Z is the sum of the number of red balls selected by Tanji and the number of heads obtained.

(c) Find the probability generating function of Z, expressing your answer as a polynomial. [3]

(d) Use the probability generating function of Z to find $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$. [5]

Q11

9231/43 • May/June 2021 • Question 4

X is a discrete random variable which takes the values $0, 2, 4, \ldots$. The probability generating function of X is given by

$$\mathrm{G}_X(t) = \frac{1}{3 - 2t^2}.$$

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [5]

(b) Find $\mathrm{P}(X = 4)$. [3]

Q12

9231/41, 9231/43 • October/November 2021 • Question 5

Nine balls labelled 1, 2, 3, 4, 5, 6, 7, 8, 9 are placed in a bag. Kai selects three balls at random from the bag, without replacement. The random variable X is the number of balls selected by Kai that are labelled with a multiple of 3.

(a) Find the probability generating function $\mathrm{G}_X(t)$ of X. [3]

The balls are replaced in the bag.

Jacob now selects two balls at random from the bag, without replacement. The random variable Y is the number of balls selected by Jacob that are labelled with an even number.

(b) Find the probability generating function $\mathrm{G}_Y(t)$ of Y. [2]

The random variable Z is the sum of the number of balls that are labelled with a multiple of 3 selected by Kai and the number of balls that are labelled with an even number selected by Jacob.

(c) Find the probability generating function of Z, expressing your answer as a polynomial. [3]

(d) Use the probability generating function of Z to find $\mathrm{E}(Z)$. [2]

Q13

9231/42 • October/November 2021 • Question 5

The random variable X is such that $P(X = r) = kr^2$ for $r = 1, 2, 3, 4$, where k is a constant.

(a) Find the value of k. [1]

(b) Find the probability generating function $G_X(t)$ of X. [2]

The random variable Y has probability generating function $G_Y(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$.

The random variable Z is the sum of X and Y.

(c) Assuming that X and Y are independent, find the probability generating function $G_Z(t)$ of Z as a polynomial in t. [3]

(d) Given that $E(Z) = \frac{13}{3}$, use $G_Z(t)$ to find $\mathrm{Var}(Z)$. [3]

Q14

9231/41, 9231/42 • May/June 2022 • Question 2

The probability generating function, $\mathrm{G}_Y(t)$, of the random variable Y is given by

$$\mathrm{G}_Y(t) = 0.04 + 0.2t + 0.37t^2 + 0.3t^3 + 0.09t^4.$$

(a) Find $\mathrm{Var}(Y)$. [4]

The random variable Y is the sum of two independent observations of the random variable X.

(b) Find the probability generating function of X, giving your answer as a polynomial in t. [3]

Q15

9231/43 • May/June 2022 • Question 3

George throws two coins, A and B, at the same time. Coin A is biased so that the probability of obtaining a head is a. Coin B is biased so that the probability of obtaining a head is b, where $b < a$. The probability generating function of X, the number of heads obtained by George, is $\mathrm{G}_X(t)$. The coefficients of t and t^2 in $\mathrm{G}_X(t)$ are $\frac{5}{12}$ and $\frac{1}{12}$ respectively.

(a) Find the value of a. [2]

The random variable Y is the sum of two independent observations of X.

(b) Find the probability generating function of Y, giving your answer as a polynomial in t. [3]

(c) Find $\mathrm{Var}(Y)$. [3]

Q16

9231/41, 9231/43 • October/November 2022 • Question 4

Jason has three biased coins. For each coin the probability of obtaining a head when it is thrown is $\frac{2}{3}$. Jason throws all three coins. The number of heads obtained is denoted by X.

(a) Find the probability generating function $G_X(t)$ of X. [3]

Jason also has two unbiased coins. He throws all five coins. The number of heads obtained from the two unbiased coins is denoted by Y. It is given that $G_Y(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$. The random variable Z is the total number of heads obtained when Jason throws all five coins.

(b) Find the probability generating function of Z, expressing your answer as a polynomial. [3]

(c) Find $\mathrm{E}(Z)$. [2]

Q17

9231/42 • October/November 2022 • Question 5

A 6-sided dice, A, with faces numbered 1, 2, 3, 4, 5, 6 is biased so that the probability of throwing a 6 is $\frac{1}{4}$. The random variable X is the number of 6s obtained when dice A is thrown twice.

(a) Find the probability generating function of X. [2]

A second dice, B, with faces numbered 1, 2, 3, 4, 5, 6 is unbiased. The random variable Y is the number of 6s obtained when dice B is thrown twice.

The random variable Z is the total number of 6s obtained when both dice are thrown twice.

(b) Find the probability generating function of Z, expressing your answer as a polynomial. [3]

(c) Find $\text{Var}(Z)$. [3]

(d) Use the probability generating function of Z to find the most probable value of Z. [1]

Q18

9231/41, 9231/42 • May/June 2023 • Question 5

Harry has three coins.

- One coin is biased so that, when it is thrown, the probability of obtaining a head is $\frac{1}{3}$.
- The second coin is biased so that, when it is thrown, the probability of obtaining a head is $\frac{1}{4}$.
- The third coin is biased so that, when it is thrown, the probability of obtaining a head is $\frac{1}{5}$.

The random variable X is the number of heads that Harry obtains when he throws all three coins together.

(a) Find the probability generating function of X. [3]

Isaac has two fair coins. The random variable Y is the number of heads that Isaac obtains when he throws both of his coins together. The random variable Z is the total number of heads obtained when Harry throws his three coins and Isaac throws his two coins.

(b) Find the probability generating function of Z, expressing your answer as a polynomial in t. [4]

(c) Use the probability generating function of Z to find $\mathrm{E}(Z)$. [2]

Q19

9231/43 • May/June 2023 • Question 5

The random variable X has probability generating function $\mathrm{G}_X(t)$ given by

$$\mathrm{G}_X(t) = k(1 + 3t + 4t^2),$$

where k is a constant.

(a) Show that $\mathrm{E}(X) = \dfrac{11}{8}$. [3]

The random variable Y has probability generating function $\mathrm{G}_Y(t)$ given by

$$\mathrm{G}_Y(t) = \tfrac{1}{3}t^2(1 + 2t).$$

The random variables X and Y are independent and $Z = X + Y$.

(b) Find the probability generating function of Z, expressing your answer as a polynomial in t. [2]

(c) Use your answer to part (b) to find the value of $\mathrm{Var}(Z)$. [3]

(d) Write down the most probable value of Z. [1]

Q20

9231/41, 9231/43 • October/November 2023 • Question 5

The random variable X has the geometric distribution $\mathrm{Geo}(p)$.

(a) Show that the probability generating function of X is $\dfrac{pt}{1-qt}$, where $q = 1 - p$. [3]

(b) Use the probability generating function of X to show that $\mathrm{Var}(X) = \dfrac{q}{p^2}$. [5]

Kenny throws an ordinary fair 6-sided dice repeatedly. The random variable X is the number of throws that Kenny takes in order to obtain a 6. The random variable Z denotes the sum of two independent values of X.

(c) Find the probability generating function of Z. [2]

Q21

9231/42 • October/November 2023 • Question 3

Toby has a bag which contains 6 red marbles and 3 green marbles. He randomly chooses 3 marbles from the bag, without replacement. The random variable X is the number of red marbles that Toby obtains.

(a) Find the probability generating function of X. [3]

Ling also has a bag which contains 6 red marbles and 3 green marbles. He randomly chooses 2 marbles from his bag, without replacement. The random variable Y is the number of red marbles that Ling obtains. It is given that the probability generating function of Y is $\frac{1}{12}(1 + 6t + 5t^2)$.

The random variable Z is the total number of red marbles that Toby and Ling obtain.

(b) Find the probability generating function of Z, expressing your answer as a polynomial in t. [3]

(c) Use the probability generating function of Z to find $\text{Var}(Z)$. [4]

Q22

9231/41, 9231/42 • May/June 2024 • Question 4

The random variable Y is the sum of two independent observations of the random variable X. The probability generating function $\mathrm{G}_Y(t)$ of Y is given by

$$\mathrm{G}_Y(t) = \frac{t^2}{(4-3t)^4}.$$

(a) Find $E(Y)$. [3]

(b) Write down an expression for the probability generating function of X. [1]

(c) Find $\mathrm{P}(X = 4)$. [3]

Q23

9231/43 • May/June 2024 • Question 4

The random variable X has probability generating function $\mathrm{G}_X(t)$ given by

$$\mathrm{G}_X(t) = ct(1+t)^5,$$

where c is a constant.

(a) Find the value of c. [1]

(b) Find the value of $E(X)$. [2]

The random variable Y is the sum of two independent values of X.

(c) Write down the probability generating function of Y and hence find $\mathrm{Var}(Y)$. [4]

(d) Find $\mathrm{P}(Y = 5)$. [2]

Q24

9231/41, 9231/43 • October/November 2024 • Question 5

Nikita has three coins. One coin is fair, one coin is biased so that the probability of obtaining a head is $\frac{1}{3}$ and the third coin is biased so that the probability of obtaining a head is $\frac{1}{5}$. The random variable X is the number of heads that Nikita obtains when he throws all three coins at the same time.

(a) Find the probability generating function of X. [3]

Rajesh has two fair six-sided dice with faces labelled 1, 2, 3, 4, 5, 6. The random variable Y is the number of 4s that Rajesh obtains when he throws the two dice.

The random variable Z is the sum of the number of heads obtained by Nikita and the number of 4s obtained by Rajesh.

(b) Find the probability generating function of Z, expressing your answer as a polynomial. [4]

(c) Use your answer to part (b) to find $\mathrm{E}(Z)$. [2]

Q25

9231/42 • October/November 2024 • Question 2

The random variable X has probability generating function $G_X(t)$ given by

$$G_X(t) = \tfrac{1}{5} + pt + qt^2,$$

where p and q are constants.

(a) Given that $\mathrm{E}(X) = 1.1$, find the numerical value of $\mathrm{Var}(X)$. [4]

The random variable Y has probability generating function $G_Y(t)$ given by

$$G_Y(t) = \tfrac{2}{3}t\left(1 + \tfrac{1}{2}t^2\right).$$

The random variable Z is the sum of independent observations of X and Y.

(b) Find the probability generating function of Z. [2]

(c) Find $\mathrm{P}(Z > 2)$. [1]

(d) State the most probable value of Z. [1]

Q26

9231/41, 9231/42 • May/June 2025 • Question 6

Y is a discrete random variable which takes the values $0, 2, 4, \ldots$. The probability generating function of Y is given by

$$G_Y(t) = \frac{k}{1 - at^2}.$$

(a) Find k in terms of a. [1]

(b) Show that $P(Y > 2) = a^2$. [3]

It is now given that $a = 0.2$.

(c) Find the value of $E(Y)$. [2]

Q27

9231/43 • May/June 2025 • Question 6

A bag contains 7 red balls and 3 blue balls. Kieran selects 2 balls at random, without replacement. The number of red balls selected by Kieran is denoted by X, and the number of different colours present in Kieran's selection is denoted by Y.

(a) Find the probability generating functions, $\mathrm{G}_X(t)$ of X and $\mathrm{G}_Y(t)$ of Y. [4]

The random variable Z is the sum of the number of red balls and the number of different colours present in Kieran's selection. Kieran claims that the probability generating function of Z is equal to $\mathrm{G}_X(t) \times \mathrm{G}_Y(t)$.

(b) Explain why Kieran is incorrect. [1]

(c) Find the probability generating function of Z, expressing your answer as a polynomial in t. [4]

(d) Use the probability generating function of Z to find $\mathrm{E}(Z)$. [2]

Q28

9231/44 • May/June 2025 • Question 5

Eric has three identical coins, each of which is biased so that the probability of obtaining a head when it is thrown is $\frac{1}{3}$. The random variable X is the number of heads obtained when Eric throws the three coins at the same time.

(a) Find the probability generating function $\mathrm{G}_X(t)$ of X. [2]

Eric also has two fair 6-sided dice with faces numbered 1 to 6. The random variable Y is the number of sixes obtained when Eric throws the two dice at the same time. It is given that the probability generating function of Y is $\frac{25}{36} + \frac{10}{36}t + \frac{1}{36}t^2$.

Eric throws the three coins and the two dice. The random variable Z is the sum of the number of heads obtained and the number of sixes obtained.

(b) Find the probability generating function $\mathrm{G}_Z(t)$ of Z, expressing your answer as a polynomial in t. [3]

(c) Use $\mathrm{G}_Z(t)$ to find $\mathrm{E}(Z)$ and $\mathrm{Var}(Z)$. [5]

Q29

9231/41, 9231/43 • October/November 2025 • Question 7

A discrete random variable X takes values $r = 0, 1, 2$ with probabilities $P(X = r)$ as given in the following table.

r	0	1	2
$P(X = r)$	a	$2a$	b

(a) Write down the probability generating function of X, and use it to find an expression for $E(X)$ in terms of a and b. [2]

(b) Show that $\text{Var}(X) = 2b + 2(a + b)(1 - 2a - 2b)$. [3]

The random variable Y is defined by $Y = X_1 + X_2 + X_3 + \cdots + X_{10}$ where $X_1, X_2, X_3, \ldots, X_{10}$ are ten independent observations of X.

(c) Using the probability generating function of Y, and your answer to part (a), show that $E(Y) = 10\,E(X)$. [3]

(d) For the case $b = 0$, define fully the distribution of Y. [2]

Q30

9231/42 • October/November 2025 • Question 6

The discrete random variable X has probability generating function $\mathrm{G}_X(t)$ given by

$$\mathrm{G}_X(t) = \frac{t}{(3-2t)^2}.$$

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [5]

The discrete random variable Y has probability generating function $\mathrm{G}_Y(t)$ given by

$$\mathrm{G}_Y(t) = \frac{t^2}{(3-2t)^2}.$$

The random variable Z is the sum of the random variables X and Y.

(b) Assuming X and Y are independent, find $\mathrm{P}(Z > 4)$. [5]

Q31

9231/44 • October/November 2025 • Question 4

The random variable X takes values 1 and 2 with probabilities $\frac{2}{5}$ and $\frac{3}{5}$ respectively.

(a) Write down the probability generating function $\mathrm{G}_X(t)$ of X. [1]

The random variable Y is the sum of four independent observations of X.

(b) Find the probability generating function $\mathrm{G}_Y(t)$ of Y. Give your answer in the form $\mathrm{G}_Y(t) = at^m(b + ct)^n$, where a, b, c, m and n are constants to be determined. [2]

(c) Use $\mathrm{G}_Y(t)$ to find $\mathrm{P}(Y = 6)$. [2]

(d) Find $\mathrm{Var}(Y)$. [5]

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Answers available in the companion mark scheme booklet.

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