Further Math with Ali Hashir

Probability Generating Functions

Topical Mark Schemes (CIE 9231)

Mark schemes corresponding to the topical questions booklet for this topic, compiled from official CIE 9231 marking schemes (2015–2025). Question numbers match the questions booklet.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Contents

Click a question number below to jump straight to its mark scheme. Every box is a live link.

For the best hyperlink experience, open this PDF in a dedicated reader – such as Adobe Acrobat, Apple Preview, or your browser's built-in PDF viewer – rather than a basic in-app or embedded preview, where links sometimes don't work.

Q1

9231/21, 9231/22 • May/June 2017 • Question 6

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Mark scheme.

Find probability of a score of 6 on no more than 4 throws: $\mathrm{P}(X \leqslant 4) = 1 - q^4$.	**M1**
Set $q = 5/6$ and evaluate: $= 671/1296$ or 0.518. **(ii)**	**A1**
Formulate condition for N ($1 - q^N$ is M0): $1 - q^{N-1} > 0.95$.	**M1**
Set $q = 5/6$, rearrange and take logs (any base) to give bound: $(5/6)^{N-1} < 0.05$, $N - 1 > \log 0.05/\log(5/6)$.	**M1**
Find $N_{\min}$ ($N - 1 < 16.4$ or $N - 1 = 16.4$ earns M1 M1 A0): $N - 1 > 16.4[3]$, $N_{\min} = 18$.	**A1**
	Total: 5

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Q2

9231/21, 9231/22 • May/June 2018 • Question 9

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Mark scheme.

Find given equation for p using $\mathrm{Var}(X) = (1-p)/p^2$: $(1-p)/p^2 = 4/9$, so $4p^2 + 9p - 9 = 0$ (AG)	**M1 A1**
Solve quadratic for p: $(4p-3)(p+3) = 0$, $p = \frac{3}{4}$ (reject $p = -3$) (ii)	**M1 A1**
$P(X = 3) = (1-p)^2 p = (\frac{1}{4})^2(\frac{3}{4}) = \frac{3}{64} = 0.0469$ (iii)	**B1**
$P(X \geq 4) = (1-p)^3 = (\frac{1}{4})^3$ or $1 - (1 + \frac{1}{4} + (\frac{1}{4})^2)\frac{3}{4} = \frac{1}{64} = 0.0156$ (iv)	**M1 A1**
Formulate condition for N: $1 - (1-p)^N > 0.999$	**M1**
$0.001 > (\frac{1}{4})^N$	**A1**
Rearrange and take logs: $N > \log 0.001 / \log 0.25$	**M1**
$N > 4.98$, so $N_{\min} = 5$	**A1**
	Total: 11

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Q3

9231/23 • May/June 2018 • Question 7

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Mark scheme.

Find given equation for p using $\mathrm{Var}(X) = (1-p)/p^2$: $(1-p)/p^2 = 3.75$, so $15p^2 + 4p - 4 = 0$ (AG)	**M1 A1**
Solve quadratic for p: $(5p-2)(3p+2) = 0$, $p = \frac{2}{5} = 0.4$ (reject $p = -\frac{2}{3}$) (ii)	**M1 A1**
$P(X = 5) = (1-p)^4 p = 0.6^4 \times 0.4 = 0.0518$ or $162/3125$ (iii)	**B1**
Find $P(3 \leq X \leq 7) = (1-p)^2 - (1-p)^7$	**M1**
$= 0.6^2 - 0.6^7 = 0.36 - 0.028 = 0.332$	**A1**
	Total: 7

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Q4

9231/21, 9231/22 • May/June 2019 • Question 6

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Mark scheme.

State mean of X: mean $= 3$ (ii)	**B1**
Find probability of score of 3 or 4 on exactly 6 throws: $P(X = 6) = q^5 p$ with $p = 1/3$, $q = 2/3$, $= 32/729$ or 0.0439 (iii)	**B1**
Find probability of score of 3 or 4 on more than 4 throws: $P(X > 4) = q^4 = 16/81$ or 0.1975 (iv)	**M1 A1**
Formulate condition for n: $1 - q^{n-1} < 0.95$	**M1**
Set $q = 2/3$, rearrange and take logs to give bound: $0.05 < (2/3)^{n-1}$, $n - 1 < \log 0.05 / \log(2/3)$	**M1**
Find $n_{\max}$: $n - 1 < 7.39$, $n_{\max} = 8$	**A1**
	Total: 7

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Q5

9231/23 • May/June 2019 • Question 7

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Mark scheme.

Find or state $E(X)$: $E(X) = 1/(1/2 \times 1/2) = 4$ (ii)	**B1**
Find probability of exactly 3 throws needed: $P(X = 3) = q^2 p$ with $p = 1/4$, $q = 3/4$, $= 9/64$ or 0.141 (iii)	**M1 A1**
Find probability of fewer than 4 throws needed: $P(X < 4) = 1 - q^3 = 37/64$ or 0.578 (iv)	**M1 A1**
Formulate condition for N: $1 - q^{N-1} > 0.95$	**M1**
Set $q = 3/4$, rearrange and take logs: $0.05 > (3/4)^{N-1}$, $N - 1 > \log 0.05 / \log 0.75$	**M1**
Find $N_{\min}$: $N - 1 > 10.4$, $N_{\min} = 12$	**A1**
	Total: 8

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Q6

9231/41, 9231/42 • May/June 2020 • Question 6

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Mark scheme.

(a)	
$P(BB) = \frac{6}{10} \cdot \frac{5}{9} = \frac{1}{3}$, $P(RB, BR) = \frac{6}{10} \cdot \frac{4}{9} \times 2 = \frac{8}{15}$, $P(RR) = \frac{4}{10} \cdot \frac{3}{9} = \frac{2}{15}$	**M1**
$G_X(t) = \frac{1}{3} + \frac{8}{15}t + \frac{2}{15}t^2$ (FT their probabilities)	**A1 FT**
(b)	
$P(1\,H) = \frac{2}{3}(1-p) + \frac{1}{3}p = \frac{7}{12}$	**M1**
$p = \frac{1}{4}$	**A1**
$G_Y(t) = \frac{1}{4} + \frac{7}{12}t + \frac{1}{6}t^2$	**A1**
(c)	
$G_Z(t) = \left(\frac{1}{3} + \frac{8}{15}t + \frac{2}{15}t^2\right)\left(\frac{1}{4} + \frac{7}{12}t + \frac{1}{6}t^2\right)$	**B1M1**
$= \frac{1}{180}\left(15 + 59t + 72t^2 + 30t^3 + 4t^4\right)$ (AEF)	**A1**
(d)	
Attempt to differentiate their $G_Z(t)$ and evaluate $G'_Z(1)$	**M1**
$E(Z) = (59 + 144 + 90 + 16)/180 = \frac{309}{180} = 1.72$	**A1**
	Total: 10

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Q7

9231/43 • May/June 2020 • Question 4

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Mark scheme.

(a)	
$\mathrm{G}_X(t) = 0.2t + 0.5t^2 + 0.3t^3$	**B1**
$\mathrm{G}_Y(t) = \left(0.2t + 0.5t^2 + 0.3t^3\right)^2$	**M1**
$= 0.04t^2 + 0.2t^3 + 0.37t^4 + 0.3t^5 + 0.09t^6$	**A1**
(b)	
$\mathrm{G}'_Y(t) = 0.08t + 0.6t^2 + 1.48t^3 + 1.5t^4 + 0.54t^5$	**M1**
$\mathrm{E}(Y) = 4.2$	**A1**
$\mathrm{G}''_Y(t) = 0.08 + 1.2t + 4.44t^2 + 6t^3 + 2.7t^4$	**M1**
Use $\mathrm{G}''_Y(1) + \mathrm{G}'_Y(1) - (\mathrm{G}'_Y(1))^2$	**M1**
$\mathrm{Var}(Y) = 14.42 + 4.2 - 4.2^2 = 0.98$	**A1**
	Total: 8

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Q8

9231/41, 9231/43 • October/November 2020 • Question 5

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Mark scheme.

B1 $\mathrm{G}_X(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$ (accept $(0.5 + 0.5t)^2$)	**(a)**
M1 A1 $\mathrm{P(0H)} = \frac{12}{36}$, $\mathrm{P(1H)} = \frac{16}{36}$, $\mathrm{P(2H)} = \frac{7}{36}$, $\mathrm{P(3H)} = \frac{1}{36}$ (attempt at probs, at least 2 correct; all correct)	**(b)**
$\mathrm{G}_Y(t) = \frac{12}{36} + \frac{16}{36}t + \frac{7}{36}t^2 + \frac{1}{36}t^3$	**B1 FT**
M1 Attempt to multiply their two PGFs	**(c)**
$\mathrm{G}_Z(t) = \frac{1}{144}\left(12 + 40t + 51t^2 + 31t^3 + 9t^4 + t^5\right)$ (obtain quintic expression and collect terms)	**M1 A1**
M1 $\mathrm{G}'_Z(t) = \frac{1}{144}\left(40 + 102t + 93t^2 + 36t^3 + 5t^4\right)$ (differentiate)	**(d)**
$\mathrm{E}(Z) = \mathrm{G}'_Z(1) = \frac{23}{12}$ $(= 1.92)$	**A1**
B1 FT 2 (power of term with largest coefficient in their $\mathrm{G}_Z(t)$)	**(e)**
	Total: 10

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Q9

9231/42 • October/November 2020 • Question 5

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Mark scheme.

B1 $\mathrm{P}(X = r) = {}^nC_r p^r (1-p)^{n-r}$ or ${}^nC_r p^r q^{n-r}$	**(a)**
$\mathrm{G}_X(t) = \sum_0^n {}^nC_r\, p^r (1-p)^{n-r} t^r$ (accept minimum of 4 terms including the last; shown with specific value of n is M0)	**M1**
$\sum_0^n {}^nC_r (pt)^r (1-p)^{n-r} = (q+pt)^n$ (at least one intermediate step shown, with p and t grouped) AG	**A1**
M1 $\mathrm{G}'_X(t) = n(q+pt)^{n-1} \times p$	**(b)**
$\mathrm{E}(X) = \mathrm{G}'_X(1) = np(q+p)^{n-1}$ and $q+p=1$, so $\mathrm{E}(X) = np$	**A1**
$\mathrm{G}''_X(t) = n(n-1)(q+pt)^{n-2} \times p \times p$	**M1**
$\mathrm{Var}(X) = n(n-1)p^2 + np - (np)^2$	**M1**
$= np(1-p)$	**A1**
	Total: 8

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Q10

9231/41, 9231/42 • May/June 2021 • Question 6

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Mark scheme.

(a)	
$\mathrm{P}(3R, 2R, 1R, 0R) = \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0$ (at least 2 correct probabilities used in a polynomial)	**M1**
$\mathrm{G}_X(t) = \frac{1}{5}t + \frac{3}{5}t^2 + \frac{1}{5}t^3$	**A1**
(b)	
$\mathrm{P}(2H, 1H, 0H) = \frac{1}{16}, \frac{6}{16}, \frac{9}{16}$ (one correct probability and all three adding to 1 used in a polynomial)	**M1**
$\mathrm{G}_Y(t) = \frac{9}{16} + \frac{6}{16}t + \frac{1}{16}t^2$	**A1**
(c)	
Attempt to multiply results from part (a) and part (b)	**M1**
Obtain polynomial	**M1**
$\frac{9}{80}t + \frac{33}{80}t^2 + \frac{28}{80}t^3 + \frac{9}{80}t^4 + \frac{1}{80}t^5$ (accept exact equivalent decimals)	**A1**
(d)	
$\mathrm{E}(Z) = \mathrm{G}'_Z(1) = \frac{1}{80}(9 + 66 + 84 + 36 + 5)$ (differentiate and put $t = 1$)	**M1**
$= \frac{200}{80} = 2.5$	**A1**
$\mathrm{G}''_Z(1) = \frac{1}{80}(66 + 168 + 108 + 20) = 4.525$ $\left[= \frac{362}{80}\right]$	**M1**
$\mathrm{Var}(Z) = \mathrm{G}''_Z(1) + \mathrm{G}'_Z(1) - (\mathrm{G}'_Z(1))^2$ (use result)	**M1**
$\frac{362}{80} + \frac{200}{80} - \left(\frac{200}{80}\right)^2 = 0.775$ $\left[= \frac{31}{40}\right]$	**A1**
	Total: 12

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Q11

9231/43 • May/June 2021 • Question 4

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Mark scheme.

(a)	
$\mathrm{G}'(t) = \dfrac{4t}{(3-2t^2)^2}$ (differentiate to obtain $\dfrac{kt}{(3-2t^2)^2}$ OE)	**M1**
so $\mathrm{E}(X) = \mathrm{G}'(1) = 4$ (CAO WWW)	**A1**
$\mathrm{G}''(t) = \dfrac{12+24t^2}{(3-2t^2)^3}$ or $4(3-2t^2)^{-2} + 32t^2(3-2t^2)^{-3}$ (OE, differentiate, allow only numerical or sign slips)	**M1**
$\mathrm{Var}(X) = \mathrm{G}''(1)+\mathrm{G}'(1)-(\mathrm{G}'(1))^2 = 36+4-16$ (substitute into correct formula, dependent on attempt at $\mathrm{G}''(t)$)	**M1**
$[\mathrm{Var}(X)] = 24$ (CAO WWW)	**A1**
(b)	
$\mathrm{G}_X(t) = \dfrac{1}{3-2t^2} = (3-2t^2)^{-1} = \dfrac{1}{3}\left(1+\dfrac{2}{3}t^2+\dfrac{4}{9}t^4+\dots\right)$ (expand given expression or give expression for the term in t^4)	**M1**
$\mathrm{P}(X=4) =$ their coefficient of t^4 (found from legitimate method)	**M1**
$[\mathrm{P}(X=4) =]\dfrac{4}{27}$ (WWW)	**A1**
	Total: 8

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Q12

9231/41, 9231/43 • October/November 2021 • Question 5

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Mark scheme.

At least 2 probabilities correct: $\mathrm{P}(3,6,9) = \frac{6}{504}$, $\mathrm{P}(\text{two of } 3,6,9) = \frac{108}{504}$, $\mathrm{P}(\text{one of } 3,6,9) = \frac{270}{504}$, $\mathrm{P}(\text{none}) = \frac{120}{504}$.	**B1**
Attempt with at least 3 probabilities in a polynomial.	**M1**
CAO.	**A1**

$$\boxed{\mathrm{G}_X(t) = \frac{20}{84} + \frac{45}{84}t + \frac{18}{84}t^2 + \frac{1}{84}t^3}$$

(b)

$\mathrm{P}(\text{both even}) = \frac{12}{72}$, $\mathrm{P}(\text{one even}) = \frac{40}{72}$, $\mathrm{P}(\text{no even}) = \frac{20}{72}$.	**M1**
CAO.	**A1**

$$\boxed{\mathrm{G}_Y(t) = \frac{5}{18} + \frac{10}{18}t + \frac{3}{18}t^2}$$

(c)

Method and attempt to multiply $\mathrm{G}_X(t) \times \mathrm{G}_Y(t)$.	**M1**
Multiplication to obtain single polynomial.	**M1**
CAO.	**A1**

$$\boxed{\mathrm{G}_Z(t) = \frac{1}{1512}\left(100 + 425t + 600t^2 + 320t^3 + 64t^4 + 3t^5\right)}$$

(d)

$\mathrm{G}'(1) = \frac{1}{1512}(425 + 1200 + 960 + 256 + 15)$.	**M1**
CWO.	**A1**
	Total: 10

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Q13

9231/42 • October/November 2021 • Question 5

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Mark scheme.

$(1+4+9+16)k = 1$, so $k = \frac{1}{30}$.	**B1**

$$\boxed{k = \frac{1}{30}}$$

(b)

Using their k in a polynomial, at least two terms correct for their k.	**M1 A1**

$$\boxed{\mathrm{G}_X(t) = \frac{1}{30}t + \frac{4}{30}t^2 + \frac{9}{30}t^3 + \frac{16}{30}t^4}$$

(c)

Method and attempt to multiply.	**M1**
Multiplication to obtain single polynomial of order 6.	**M1 A1**

$$\boxed{\mathrm{G}_Z(t) = \frac{1}{120}\left(t + 6t^2 + 18t^3 + 38t^4 + 41t^5 + 16t^6\right)}$$

(d)

Differentiate twice: $\mathrm{G}''(t) = \frac{1}{120}(12 + 108t + 456t^2 + 820t^3 + 480t^4)$; use $\mathrm{G}'(1) = \frac{13}{3}$.	**M1**
Use correct formula: $\mathrm{Var}(X) = \mathrm{G}''(1) + \frac{13}{3} - \left(\frac{13}{3}\right)^2$.	**M1**
CAO.	**A1**
	Total: 9

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Q14

9231/41, 9231/42 • May/June 2022 • Question 2

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Mark scheme.

$G'_Y(t) = 0.2 + 0.74t + 0.9t^2 + 0.36t^3$, $G''_Y(t) = 0.74 + 1.8t + 1.08t^2$ (PGF differentiated twice).	**M1**
$G'_Y(1) = 2.2$, $G''_Y(1) = 3.62$.	**A1**
$\text{Var}(Y) = G''_Y(1) + G'_Y(1) - (G'_Y(1))^2 = 3.62 + 2.2 - 2.2^2$ (correct formula used and attempt at $\text{Var}(Y)$).	**M1**
0.98. **(b)**	**A1**
$G_Y(t) = (a + bt + ct^2)^2$ (correct method).	**M1***
$a^2 = 0.04,\ a = 0.2;\ c^2 = 0.09,\ c = 0.3;\ 2ab = 0.2,\ b = \dfrac{0.1}{0.2} = 0.5$ (attempt to find all coefficients).	**depM1**
$G_X(t) = 0.2 + 0.5t + 0.3t^2$.	**A1**
	Total: 7

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Q15

9231/43 • May/June 2022 • Question 3

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Mark scheme.

$G_X(t) = \frac{6}{12} + \frac{5}{12}t + \frac{1}{12}t^2$; P(0 heads) $= (1-a)(1-b) = \frac{6}{12}$, P(1 head) $= a(1-b) + (1-a)b = \frac{5}{12}$, P(2 heads) $= ab = \frac{1}{12}$, giving $a + b = \frac{7}{12}$, $ab = \frac{1}{12}$ (obtain two correct equations and attempt to solve).	**M1**
$a = \frac{1}{3}$ $(b = \frac{1}{4})$ (correct value for a). **(b)**	**A1**
$G_Y(t) = \left(\frac{6}{12} + \frac{5}{12}t + \frac{1}{12}t^2\right)^2$ (square their $G_X(t)$).	**M1**
Obtain quartic polynomial from 3-term $G_X(t)$.	**M1**
$\frac{1}{144}\left(36 + 60t + 37t^2 + 10t^3 + t^4\right)$. **(c)**	**A1**
$G'_Y(t) = \frac{1}{144}(60 + 74t + 30t^2 + 4t^3)$, $G''_Y(t) = \frac{1}{144}(74 + 60t + 12t^2)$ (differentiate their $G_Y(t)$ twice).	**M1**
$\mathrm{Var}(Y) = G''_Y(1) + G'_Y(1) - (G'_Y(1))^2 = \frac{1}{144}(146) + \frac{7}{6} - \frac{49}{36} = \frac{59}{72}$ (use correct formula and attempt at $\mathrm{Var}(Y)$).	**M1**
0.819.	**A1**
	Total: 8

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Q16

9231/41, 9231/43 • October/November 2022 • Question 4

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Mark scheme.

$P(0H)=\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{27}$, $P(1H)=\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\times 3=\frac{6}{27}$, $P(2H)=\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\times 3=\frac{12}{27}$, $P(3H)=\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}=\frac{8}{27}$ (at least 2 correct)	**M1**
$G_X(t)=\frac{1}{27}+\frac{6}{27}t+\frac{12}{27}t^2+\frac{8}{27}t^3$ (correct form, ft their probabilities)	**M1 A1**
OR: PGF for one coin is $\frac{1}{3}+\frac{2}{3}t$ (M1); for 3 coins, PGF is $\left(\frac{1}{3}+\frac{2}{3}t\right)^3$ (M1A1) **(b)**	
$G_Z(t)=\frac{1}{108}(1+2t+t^2)(1+6t+12t^2+8t^3)$ (correct method)	**M1**
$=\frac{1}{108}(1+8t+25t^2+38t^3+28t^4+8t^5)$ (obtain quintic polynomial)	**M1**
$\frac{1}{108}+\frac{2}{27}t+\frac{25}{108}t^2+\frac{19}{54}t^3+\frac{7}{27}t^4+\frac{2}{27}t^5$ **(c)**	**A1**
$G'_Z(t)=\frac{1}{108}(8+50t+114t^2+112t^3+40t^4)$; $\mathrm{E}(Z)=\frac{1}{108}(8+50+114+112+40)$ (differentiate and evaluate at $t=1$)	**M1**
$\frac{324}{108}=3$	**A1**
Alternative: M1 summing expected values for each coin $\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{1}{2}+\frac{1}{2}$; A1 $=3$	
	Total: 8

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Q17

9231/42 • October/November 2022 • Question 5

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Mark scheme.

$G_X(t) = \frac{9}{16} + \frac{6}{16}t + \frac{1}{16}t^2$ (2 probabilities correct, in a polynomial) **(b)**	**M1 A1**
$G_Z(t) = \left(\frac{9}{16} + \frac{6}{16}t + \frac{1}{16}t^2\right)\left(\frac{25}{36} + \frac{10}{36}t + \frac{1}{36}t^2\right)$ (second PGF correct and multiplied by part (a))	**M1**
$= \frac{1}{576}(225 + 240t + 94t^2 + 16t^3 + t^4)$ (obtains a quartic polynomial)	**M1**
$\frac{25}{64} + \frac{5}{12}t + \frac{47}{288}t^2 + \frac{1}{36}t^3 + \frac{1}{576}t^4$ **(c)**	**A1**
$G'_Z(t) = \frac{1}{576}(240+188t+48t^2+4t^3)$, $G''_Z(t) = \frac{1}{576}(188+96t+12t^3)$ (differentiate twice)	**M1**
$\mathrm{Var}(Z) = \frac{1}{576}(188 + 96 + 12) + \frac{5}{6} - \frac{25}{36}$ (use correct formula)	**M1**
$\frac{47}{72}$	**A1**
OR: $\mathrm{Var}(Z) = 2 \times \frac{1}{4} \times \frac{3}{4} + 2 \times \frac{1}{6} \times \frac{5}{6}$ (M1 one term correct, M1 two terms added), A1 $\frac{47}{72}$ **(d)**	
$Z = 1$ (FT their power with greatest coefficient in part (b))	**B1 FT**
	Total: 9

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Q18

9231/41, 9231/42 • May/June 2023 • Question 5

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Mark scheme.

$\mathrm{P}(3H) = \frac{1}{60}$, $\mathrm{P}(2H) = \frac{9}{60}$, $\mathrm{P}(1H) = \frac{26}{60}$, $\mathrm{P}(0H) = \frac{24}{60}$. Two probabilities correct, seen anywhere.	**B1**
$G_X(t) = \frac{24}{60} + \frac{26}{60}t + \frac{9}{60}t^2 + \frac{1}{60}t^3$. Cubic polynomial with 4 probabilities as coefficients with the correct powers of t from their working. Equivalent forms are acceptable.	**M1**
Correct, AEF. **(b)**	**A1**
$G_Y(t) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$.	**B1**
$G_Z(t) = \left(\frac{24}{60} + \frac{26}{60}t + \frac{9}{60}t^2 + \frac{1}{60}t^3\right)\left(\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2\right)$. Attempt to multiply their two PGFs.	**M1**
$\frac{1}{240}\left(24 + 74t + 85t^2 + 45t^3 + 11t^4 + t^5\right)$. Obtain quintic expression and collect terms.	**M1**
Correct, AEF. **(c)**	**A1**
Differentiate: $G'_Z(t) = \frac{1}{240}\left(74 + 170t + 135t^2 + 44t^3 + 5t^4\right)$. Differentiate their G.	**M1**
$G'_Z(1) = \frac{428}{240} = \frac{107}{60} = 1.78$. Any correct form.	**A1**
	Total: 9

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Q19

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Mark scheme.

$[G_X(t) = k(1+3t+4t^2)]$ $k(1+3+4)=1$, $k=\frac{1}{8}$. Working required.	**B1**
$G'_X(t) = k(3+8t)$. Or $\sum px = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{4}{8}$ $[= \frac{11}{8}]$.	**M1**
$E(X) = G'_X(1) = \frac{11}{8}$ (AG). Evidence of using $t=1$ or $\sum px$ required. CWO. **(b)**	**A1**
$G_Z(t) = \frac{1}{8}(1+3t+4t^2) \times \frac{1}{3}t^2(1+2t)$. Multiply the two PGFs to obtain a single polynomial of degree 5.	**M1**
$\frac{1}{24}\left(t^2+5t^3+10t^4+8t^5\right)$. May have t^2 as a factor. **(c)**	**A1**
$G'_Z(t) = \frac{1}{24}\left(2t+15t^2+40t^3+40t^4\right)$, $G''_Z(t) = \frac{1}{24}\left(2+30t+120t^2+160t^3\right)$. Differentiate twice, allow one slip.	**M1**
$\mathrm{Var}(X) = G''_Z(1) + G'_Z(1) - (G'_Z(1))^2 = \frac{1}{24}(312) + \frac{97}{24} - \left(\frac{97}{24}\right)^2$. Use correct formula.	**M1**
0.707 $(= \frac{407}{576})$. **(d)**	**A1**
$(Z=)4$. FT their final polynomial in part (b).	**B1 FT**
	Total: 9

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Q20

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Mark scheme.

$\mathrm{P}(X = r) = p(1-p)^{r-1} = pq^{r-1}$ (implied by $\mathrm{G}(t) = \sum pq^{r-1}t^r$).	**B1**
$\mathrm{G}(t) = \sum pt(qt)^{r-1}$ (infinite summation with common ratio qt and first term/common factor pt indicated, or GP identified).	**M1**
$\mathrm{G}(t) = pt\left(\dfrac{1}{1-qt}\right) = \dfrac{pt}{1-qt}$ (AG, convincingly obtained). For part (b):	**A1**
$\mathrm{G}'(t) = pqt(1-qt)^{-2} + p(1-qt)^{-1} = p(1-qt)^{-2}$ (attempt to differentiate as product/quotient).	**M1**
$\mathrm{G}''(t) = 2pq^2t(1-qt)^{-3} + 2pq(1-qt)^{-2} = 2qp(1-qt)^{-3}$ (attempt to differentiate their $\mathrm{G}'(t)$).	**M1**
$\mathrm{G}'(1) = p(1-q)^{-2} = \frac{1}{p}$, $\mathrm{G}''(1) = \frac{2q}{p^2}$ (substitute $t = 1$ and use the variance formula).	**M1**
$\mathrm{Var}(X) = \frac{2q}{p^2} + \frac{1}{p} - \left(\frac{1}{p}\right)^2$ ($(1-q)$ replaced by p throughout, cancellation seen).	**M1**
$\mathrm{Var}(X) = \dfrac{q}{p^2}$ (AG, convincingly obtained). For part (c):	**A1**
$\mathrm{G}_X(t) = \dfrac{\frac{1}{6}t}{1-\frac{5}{6}t} = \dfrac{t}{6-5t}$ (implied by correct final answer).	**B1**
$\mathrm{G}_Z(t) = (\mathrm{G}_X(t))^2 = \left(\dfrac{t}{6-5t}\right)^2$.	**B1**
	Total: 10

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Q21

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Mark scheme.

$\mathrm{P}(3R) = \frac{120}{504}$, $\mathrm{P}(2R) = \frac{270}{504}$, $\mathrm{P}(1R) = \frac{108}{504}$, $\mathrm{P}(0R) = \frac{6}{504}$ (2 correct probabilities).	**B1**
$\mathrm{G}_X(t) = \frac{1}{504}(6 + 108t + 270t^2 + 120t^3) = \frac{1}{84}(1 + 18t + 45t^2 + 20t^3)$ (cubic polynomial with their probabilities).	**M1**
$\mathrm{G}_X(t) = \frac{1}{84} + \frac{3}{14}t + \frac{15}{28}t^2 + \frac{5}{21}t^3$. For part (b):	**A1**
$\mathrm{G}_Z(t) = \frac{1}{84}(1 + 18t + 45t^2 + 20t^3) \times \frac{1}{12}(1 + 6t + 5t^2)$ (attempt to multiply out).	**M1**
$= \frac{1}{1008}(1 + 24t + 158t^2 + 380t^3 + 345t^4 + 100t^5)$ (obtain quintic polynomial).	**M1**
$\mathrm{G}_Z(t) = \frac{1}{1008} + \frac{1}{42}t + \frac{79}{504}t^2 + \frac{95}{252}t^3 + \frac{115}{336}t^4 + \frac{25}{252}t^5$. For part (c):	**A1**
$\mathrm{G}' = \frac{1}{1008}(24 + 316t + 1140t^2 + 1380t^3 + 500t^4)$, $\mathrm{G}'' = \frac{1}{1008}(316 + 2280t + 4140t^2 + 2000t^3)$ (differentiate twice).	**M1**
$\mathrm{E}(Z) = \frac{3360}{1008} = \frac{10}{3}$ (seen or implied, NFWW).	**B1**
$\mathrm{Var}(Z) = \frac{8736}{1008} + \frac{10}{3} - \left(\frac{10}{3}\right)^2 = \frac{26}{3} - \frac{10}{3} - \frac{100}{9}$ (use formula with their $\mathrm{G}''(1)$ and $\mathrm{G}'(1)$).	**M1**
$\mathrm{Var}(Z) = \frac{8}{9}$ (or 0.889).	**A1**
	Total: 10

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Q22

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Mark scheme.

$\mathrm{G}_Y(t) = \dfrac{t^2}{(4-3t)^4}$, $\mathrm{G}'_Y(t) = \dfrac{(4-3t)^4(2t) - (-12)t^2(4-3t)^3}{(4-3t)^8}$ or $\dfrac{2t}{(4-3t)^4} + \dfrac{12t^2}{(4-3t)^5}$. 'Two' term obtained, correct denominator(s).	**M1**
$E(Y) = \mathrm{G}'_Y(1) = 14$. $t = 1$ in their expression. CWO. (b)	**M1 A1**
$\mathrm{G}_X(t) = \dfrac{t}{(4-3t)^2}$. (c)	**B1**
$\mathrm{G}_X(t) = t(4-3t)^{-2} = \frac{t}{16}\left(1 - \frac{3}{4}t\right)^{-2} = \frac{t}{16}\left(1 + \frac{3}{2}t - 3 \times \frac{9}{16}t^2 + 4 \times \frac{27}{64}t^3 + \ldots\right)$. Expansion as far as t^3 OR term in t^4 calculated.	**M1**
Use $P(X = 4) =$ their coefficient of t^4. Find the numerical value of their coefficient of t^4.	**M1**
$\dfrac{27}{256}$ (accept 0.105).	**A1**
	Total: 7

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Q23

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Mark scheme.

$c = \dfrac{1}{32}$ (accept 0.03125 or 0.0313). (b)	**B1**
$\mathrm{G}_X(t) = ct(1+t)^5$, $\mathrm{G}'_X(t) = c\left(5t(1+t)^4 + (1+t)^5\right)$. Must use product rule OR differentiate their expansion of G. Allow one error in their differentiation. Condone missing c.	**M1**
$E(X) = \mathrm{G}'_X(1) = c(80+32) = 3.5$. Correct work only. (c)	**A1**
$\mathrm{G}_Y(t) = c^2t^2(1+t)^{10}$.	***M1**
$\mathrm{G}'_Y(t) = c^2\left(10t^2(1+t)^9 + 2t(1+t)^{10}\right)$, $\mathrm{G}''_Y(t) = c^2\left(90t^2(1+t)^8 + 40t(1+t)^9 + 2(1+t)^{10}\right)$. Differentiate their $\mathrm{G}_Y(t)$ twice, allow one error.	**DM1**
$\mathrm{G}'_Y(1) = 7$, $\mathrm{G}''_Y(1) = 44.5$; $\mathrm{Var}(Y) = \mathrm{G}''_Y(1) + \mathrm{G}'_Y(1) - (\mathrm{G}'_Y(1))^2 = 2.5$. Use correct formula and substitute values for $t = 1$ (may be in terms of c). A1 for correct work only. (d)	**M1 A1**
$\mathrm{P}(Y = 5) =$ coefficient of t^5 in expansion of their $c^2t^2(1+t)^{10}$; required term is $c^2 \times 120$; expansion $= c^2t^2\left(1 + 10t + 45t^2 + 120t^3 + \ldots\right)$. Identify required term.	**M1**
$\dfrac{15}{128}$ (0.117, $\frac{120}{1024}$, or any equivalent fraction).	**A1**
	Total: 9

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Q24

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Mark scheme.

(a)	
$\mathrm{P}(0 \text{ heads}) = \frac{8}{30}$, $\mathrm{P}(1 \text{ head}) = \frac{14}{30}$, $\mathrm{P}(2 \text{ heads}) = \frac{7}{30}$, $\mathrm{P}(3 \text{ heads}) = \frac{1}{30}$ – all correct.	**B1**
$G_X(t) = \frac{8}{30} + \frac{14}{30}t + \frac{7}{30}t^2 + \frac{1}{30}t^3$ – cubic polynomial.	**M1**
FT their probabilities that sum to one.	**A1 FT**
(b)	
$G_Y(t) = \frac{25}{36} + \frac{10}{36}t + \frac{1}{36}t^2$.	**B1**
$G_Z(t) = G_X(t)\,G_Y(t)$ with attempt to multiply.	**M1**
Obtain quintic polynomial $G_Z(t) = \frac{1}{1080}\left(200 + 430t + 323t^2 + 109t^3 + 17t^4 + t^5\right)$.	**M1 A1**
(c)	
$G'_Z(1) = \frac{1}{1080}(430 + 646 + 327 + 68 + 5)$ – differentiate and substitute $t = 1$.	**M1**
$\frac{1476}{1080} = \frac{41}{30} = 1.37$.	**A1**
	Total: 9

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Q25

9231/42 • October/November 2024 • Question 2

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Mark scheme.

(a)	
$G_X(t) = \frac{1}{5} + pt + qt^2$, $G'_X(t) = p + 2qt$, so $p + 2q = 1.1$.	**B1**
$\frac{1}{5} + p + q = 1 \Rightarrow p + q = \frac{4}{5}$, and solve.	**M1**
$p = \frac{1}{2}$, $q = \frac{3}{10}$.	**A1**
$G''_X(t) = 2q = \frac{3}{5}$, $\mathrm{Var}(X) = 0.6 + 1.1 - 1.1^2 = 0.49$.	**A1**
(b)	
$G_Z(t) = \left(\frac{1}{5} + \frac{1}{2}t + \frac{3}{10}t^2\right) \times \frac{2}{3}t\left(1 + \frac{1}{2}t^2\right)$ (wrong/swapped p, q or missing t: M1A0).	**M1**
$\dfrac{1}{30}\left(4t + 10t^2 + 8t^3 + 5t^4 + 3t^5\right)$ (any equivalent form).	**A1**
(c)	
$\dfrac{16}{30} = \dfrac{8}{15}$ (FT their fully expanded PGF).	**B1 FT**
(d)	
$Z = 2$ (correct work only, from correct expanded polynomial).	**B1**
	Total: 8

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Q26

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Mark scheme.

B1 $k = 1 - a$.	**(a)**
*M1 Attempt at series for $(1 - at^2)^{-1} = 1 + at^2 + \dots$	**(b)**
DM1 $\mathrm{P}(Y > 2) = 1 - k(1 + a)$ (correct un-simplified expression).	
A1 $\mathrm{P}(Y > 2) = 1 - (1 - a)(1 + a) = a^2$ (AG, shown convincingly).	
M1 Attempt to differentiate $\mathrm{G}_Y(t)$ with their k: $\mathrm{G}'_Y(t) = \dfrac{0.32t}{(1 - 0.2t^2)^2}$.	**(c)**
A1 $\mathrm{E}(Y) = \mathrm{G}'_Y(1) = \dfrac{0.32}{(1 - 0.2)^2} = 0.5$.	
	Total: 6

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Q27

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Mark scheme.

B1 Probabilities $\frac{1}{15}, \frac{7}{15}, \frac{7}{15}$ for 0, 1, 2 red balls.	**(a)**
B1 FT $\mathrm{G}_X(t) = \frac{1}{15} + \frac{7}{15}t + \frac{7}{15}t^2$.	
B1 Probabilities $\frac{8}{15}, \frac{7}{15}$ for 1, 2 colours.	
B1 FT $\mathrm{G}_Y(t) = \frac{8}{15}t + \frac{7}{15}t^2$.	
B1 X and Y are not independent.	**(b)**
B1 Correct 3 cases identified where probabilities are non-zero: $(x, y) = (0,1), (1,2), (2,1)$, giving $z = 1, 3, 3$.	**(c)**
M1 At least one non-zero probability correct (allow $\frac{1}{15}, \frac{7}{15}, \frac{14}{15}$).	
A1 All probabilities correct: $P(Z = 1) = \frac{1}{15}$, $P(Z = 3) = \frac{7}{15} + \frac{7}{15} = \frac{14}{15}$.	
A1 FT $\mathrm{G}_Z(t) = \frac{1}{15}t + \frac{14}{15}t^3$.	
M1 Attempt to differentiate $\mathrm{G}_Z(t)$ and find $\mathrm{G}'_Z(1)$: $\mathrm{G}'_Z(t) = \frac{1}{15} + \frac{42}{15}t^2$.	**(d)**
A1 $\mathrm{G}'_Z(1) = \frac{43}{15} = 2.87$.	
	Total: 11

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Q28

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Mark scheme.

B1 Probabilities: $\frac{8}{27}, \frac{12}{27}, \frac{6}{27}, \frac{1}{27}$ for 0, 1, 2, 3 heads.	**(a)**
B1 FT $\mathrm{G}_X(t) = \frac{8}{27} + \frac{12}{27}t + \frac{6}{27}t^2 + \frac{1}{27}t^3$.	
M1 Attempt to multiply their $\mathrm{G}_X(t)$ and their $\mathrm{G}_Y(t)$.	**(b)**
M1 Expand to form polynomial of degree 5.	
A1 $\mathrm{G}_Z(t) = \frac{1}{972}(200 + 380t + 278t^2 + 97t^3 + 16t^4 + t^5)$ (CWO).	
M1 Attempt at differentiating their $\mathrm{G}_Z(t)$: $\mathrm{G}'_Z(t) = \frac{1}{972}(380 + 556t + 291t^2 + 64t^3 + 5t^4)$.	**(c)**
A1 FT $\mathrm{E}(Z) = \mathrm{G}'_Z(1) = \frac{1296}{972} = \frac{4}{3}$.	
M1 Attempt at differentiating their $\mathrm{G}'_Z(t)$: $\mathrm{G}''_Z(t) = \frac{1}{972}(556 + 582t + 192t^2 + 20t^3)$.	
M1 Use correct formula: $\mathrm{Var}(Z) = \frac{1350}{972} + \frac{4}{3} - (\frac{4}{3})^2$.	
A1 $= \frac{17}{18} = 0.944$ (CWO).	
	Total: 10

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Q29

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Mark scheme.

$\mathrm{G}_X(t) = a + 2at + bt^2$.	**B1**
$\mathrm{G}'_X(t) = 2a + 2bt$, $\mathrm{E}(X) = \mathrm{G}'_X(1) = 2a + 2b$. (b)	**B1**
$\mathrm{G}''_X(t) = 2b$.	**B1**
$\mathrm{Var}(X) = 2b + (2a + 2b) - (2a + 2b)^2$.	**M1**
$= 2b + (2a + 2b)(1 - (2a + 2b)) = 2b + 2(a + b)(1 - 2a - 2b)$ (AG, shown convincingly). (c)	**A1**
$\mathrm{G}_Y(t) = (a + 2at + bt^2)^{10}$.	**B1**
Attempt to differentiate their $\mathrm{G}_Y(t)$ and evaluate at $t = 1$: $\mathrm{G}'_Y(t) = 20(a + bt)(a + 2at + bt^2)^9$, $\mathrm{E}(Y) = 20(a + b)(3a + b)^9$.	**M1**
$3a + b = 1$ since total probability is 1, hence $\mathrm{E}(Y) = 20(a + b) = 10\mathrm{E}(X)$. (d)	**A1**
Uses PGF to identify binomial.	***B1**
Identifies binomial with correct parameters: $\mathrm{B}\left(10, \frac{2}{3}\right)$.	**DB1**
	Total: 10

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Q30

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Mark scheme.

Attempt to differentiate $\mathrm{G}_X(t)$: $\mathrm{G}'_X(t) = \dfrac{3+2t}{(3-2t)^3}$.	**M1**
$\mathrm{E}(X) = \mathrm{G}'_X(1) = 5$.	**A1**
Attempt to differentiate $\mathrm{G}'_X(t)$: $\mathrm{G}''_X(t) = \dfrac{24+8t}{(3-2t)^4}$.	**M1**
Use correct formula: $\mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - (\mathrm{G}'_X(1))^2 = 32 + 5 - 25$.	**M1**
$\mathrm{Var}(X) = 12$ (CWO). (b)	**A1**
Multiply and attempt at binomial expansion: $\mathrm{G}_Z(t) = \dfrac{t^3}{(3-2t)^4} = \frac{1}{81}t^3\left(1 + (-4)(-\frac{2}{3}t) + \cdots\right)$.	**M1**
Attempt at finding the coefficient of t^3 or t^4.	**M1**
$\mathrm{P}(Z=4) = \frac{8}{243}$ and $\mathrm{P}(Z=3) = \frac{1}{81}$.	**A1**
Attempt to find $\mathrm{P}(Z>4)$ using their values.	**M1**
$\mathrm{P}(Z>4) = \frac{232}{243}$ (AWRT 0.955).	**A1**
	Total: 10

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Mark scheme.

$\mathrm{G}_X(t) = \frac{2}{5}t + \frac{3}{5}t^2$ (CAO). (b)	**B1**
Attempt at 4th power of their $\mathrm{G}_X(t)$.	**M1**
$\mathrm{G}_Y(t) = \frac{1}{625}t^4(2+3t)^4$. (c)	**A1**
Attempt at coefficient of t^6 in their $\mathrm{G}_Y(t)$.	**M1**
$\mathrm{P}(Y=6) = \frac{216}{625}$ (AWRT 0.346). (d)	**A1**
Attempt at differentiating their $\mathrm{G}_Y(t)$.	**M1**
Attempt at differentiating their $\mathrm{G}'_Y(t)$.	**M1**
$\mathrm{G}'_Y(1) = \frac{32}{5} = 6.4$, $\mathrm{G}''_Y(1) = \frac{888}{25} = 35.52$ (WWW).	**B1**
Use correct formula: $\mathrm{Var}(Y) = \frac{888}{25} + \frac{32}{5} - \left(\frac{32}{5}\right)^2$.	**M1**
$\mathrm{Var}(Y) = \frac{24}{25}$ $(= 0.96)$ (CWO).	**A1**
	Total: 10

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Written by Ali Hashir.

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