Further Math with Ali Hashir

Linear Motion Under a Variable Force

Topical Questions (CIE 9231)

Compiled from CIE 9231 Further Mathematics past papers (2015–2025), sorted by topic. Each question is numbered for this booklet, with its original paper code and session shown beneath.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Q1

9231/21, 9231/22 • May/June 2015 • Question 2

A particle P moves on a straight line AOB in simple harmonic motion. The centre of the motion is O, and P is instantaneously at rest at A and B. The point C is on the line AOB, between A and O, and $CO = 10$ m. When P is at C, the magnitude of its acceleration is 0.625 m s^{-2} and it is moving towards O with speed 6 m s^{-1}. Find

(i) the period of the motion, in terms of π, [2]

(ii) the amplitude of the motion. [2]

The point M is the mid-point of OB. Find the time that P takes to travel directly from C to M. [4]

Q2

9231/23 • May/June 2015 • Question 3

A particle moves on a straight line AOB in simple harmonic motion, where $AB = 2a$ m. The centre of the motion is O and the particle is instantaneously at rest at A and B. The point M is the mid-point of OB. The particle passes through M moving towards O and next achieves its maximum speed one second later. Find the period of the motion. [4]

Find the distance of the particle from O when its speed is equal to one half of its maximum speed. [3]

At an instant 2.5 seconds after the particle passes through M moving towards O, the distance of the particle from O is $\sqrt{2}$ m. Find, in metres, the amplitude of the motion. [2]

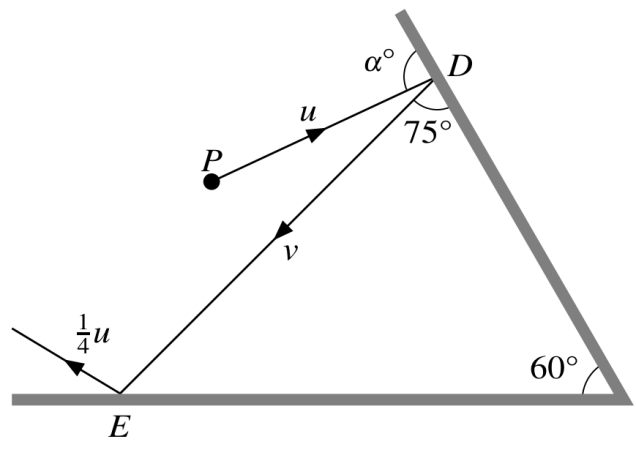
Q3

9231/21, 9231/22 • May/June 2016 • Question 3

A particle P is performing simple harmonic motion with amplitude 0.25 m. During each complete oscillation, P moves with a speed that is less than or equal to half of its maximum speed for $\frac{4}{3}$ seconds. Find the period of the motion and the maximum speed of P. [8]

Q4

9231/21, 9231/22, 9231/23 • October/November 2016 • Question 1



The point C is on the fixed line l. Points A and B on l are such that $AC = 4$ m and $CB = 2$ m, with C between A and B. The point M is the mid-point of AB (see diagram). A particle P of mass m oscillates between A and B in simple harmonic motion. When P is at C, its speed is 4 m s^{-1}. Find

(i) the magnitude of the maximum acceleration of P, [3]

(ii) the number of complete oscillations made by P in one minute, [2]

(iii) the time that P takes to travel directly from A to C. [3]

Q5

9231/21, 9231/22, 9231/23 • October/November 2017 • Question 2

The piston in a large engine rises and falls in simple harmonic motion. When the piston is 1.6 m below its highest level, the rate of change of its height is $\frac{3}{5}\pi$ metres per second. When the piston is 0.2 m below its highest level, the rate of change of its height is $\frac{1}{4}\pi$ metres per second. Find the amplitude and period of the motion. [7]

Q6

9231/21, 9231/23 • October/November 2018 • Question 1

A particle P oscillates in simple harmonic motion between the points A and B, where $AB = 6$ m. The period of the motion is $\frac{1}{2}\pi$ s. Find the speed of P when it is 2 m from B. [3]

Q7

9231/21, 9231/23 • October/November 2018 • Question 5

The fixed points A and B are on a smooth horizontal surface with $AB = 2.6$ m. One end of a light elastic spring, of natural length 1.25 m and modulus of elasticity λ N, is attached to A. The other end is attached to a particle P of mass 0.4 kg. One end of a second light elastic spring, of natural length 1.0 m and modulus of elasticity 0.6λ N, is attached to B; its other end is attached to P. The system is in equilibrium with P on the surface at the point E.

(i) Show that $AE = 1.4$ m. [4]

The particle P is now displaced slightly from E, along the line AB.

(ii) Show that, in the subsequent motion, P performs simple harmonic motion. [5]

(iii) Given that the period of the motion is $\frac{1}{7}\pi$ s, find the value of λ. [3]

Q8

9231/22 • October/November 2018 • Question 1

The point O is on the fixed horizontal line l. Points A and B on l are such that $OA = 0.1$ m and $OB = 0.5$ m, with A between O and B. A particle P oscillates on l in simple harmonic motion with centre O. The kinetic energy of P when it is at A is twice its kinetic energy when it is at B. Find the amplitude of the motion. [3]

Q9

9231/21, 9231/22 • May/June 2019 • Question 2

A particle P moves on a straight line in simple harmonic motion. The centre of the motion is O. The points A and B are on the line on opposite sides of O such that $OA = 3.5\,\text{m}$ and $OB = 1\,\text{m}$. The speed of P when it is at B is twice its speed when it is at A. The maximum acceleration of P is $1\,\text{m}\,\text{s}^{-2}$.

(i) Find the speed of P when it is at O. [4]

(ii) Find the time taken by P to travel directly from A to B. [4]

Q10

9231/31, 9231/32 • May/June 2020 • Question 5

A particle P is moving along a straight line with acceleration $3ku - kv$ where v is its velocity at time t, u is its initial velocity and k is a constant. The velocity and acceleration of P are both in the direction of increasing displacement from the initial position.

(a) Find the time taken for P to achieve a velocity of $2u$. [3]

(b) Find an expression for the displacement of P from its initial position when its velocity is $2u$. [5]

Q11

9231/33 • May/June 2020 • Question 2

A particle Q of mass m kg falls from rest under gravity. The motion of Q is resisted by a force of magnitude mkv N, where v m s^{-1} is the speed of Q at time t s and k is a positive constant.

Find an expression for v in terms of g, k and t. [6]

Q12

9231/31, 9231/33 • October/November 2020 • Question 7

A particle P moving in a straight line has displacement x m from a fixed point O on the line at time t s. The acceleration of P, in m s^{-2}, is given by $\frac{200}{x^2} - \frac{100}{x^3}$ for $x > 0$. When $t = 0$, $x = 1$ and P has velocity 10 m s^{-1} directed towards O.

(a) Show that the velocity v m s^{-1} of P is given by $v = \frac{10(1-2x)}{x}$. [5]

(b) Show that x and t are related by the equation $e^{-40t} = (2x-1)e^{2x-2}$ and deduce what happens to x as t becomes large. [5]

Q13

9231/32 • October/November 2020 • Question 7

A particle P of mass m kg moves in a horizontal straight line against a resistive force of magnitude mkv^2 N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P after it has moved a distance x m and k is a positive constant. The initial speed of P is $u\,\mathrm{m\,s^{-1}}$.

(a) Show that $x = \frac{1}{k}\ln 2$ when $v = \frac{1}{2}u$. [4]

Beginning at the instant when the speed of P is $\frac{1}{2}u$, an additional force acts on P. This force has magnitude $\frac{5m}{v}$ N and acts in the direction of increasing x.

(b) Show that when the speed of P has increased again to $u\,\mathrm{m\,s^{-1}}$, the total distance travelled by P is given by an expression of the form

$$\frac{1}{3k}\ln\left(\frac{A-ku^3}{B-ku^3}\right),$$

stating the values of the constants A and B. [7]

Q14

9231/31, 9231/32 • May/June 2021 • Question 1

A particle P of mass 1 kg is moving along a straight line against a resistive force of magnitude $\frac{10\sqrt{v}}{(t+1)^2}$ N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P at time t s. When $t = 0$, $v = 25$.

Find an expression for v in terms of t. [5]

Q15

9231/33 • May/June 2021 • Question 5

A particle P of mass m kg is projected vertically upwards from a point O, with speed $20\,\mathrm{m\,s^{-1}}$, and moves under gravity. There is a resistive force of magnitude $2mv$ N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P at time t s after projection.

(a) Find an expression for v in terms of t, while P is moving upwards. [6]

The displacement of P from O is x m at time t s.

(b) Find an expression for x in terms of t, while P is moving upwards. [2]

(c) Find, correct to 3 significant figures, the greatest height above O reached by P. [2]

Q16

9231/31, 9231/33 • October/November 2021 • Question 2

A particle P of mass m kg moves along a horizontal straight line with acceleration a m s^{-2} given by

$$a = \frac{v(1 - 2t^2)}{t},$$

where v m s^{-1} is the velocity of P at time t s.

(a) Find an expression for v in terms of t and an arbitrary constant. [3]

(b) Given that $a = 5$ when $t = 1$, find an expression, in terms of m and t, for the horizontal force acting on P at time t. [3]

Q17

9231/32 • October/November 2021 • Question 6

A particle P of mass 2 kg moves along a horizontal straight line. The point O is a fixed point on this line. At time t s the velocity of P is v m s^{-1} and the displacement of P from O is x m.

A force of magnitude $\left(8x - \frac{128}{x^3}\right)$ N acts on P in the direction OP. When $t = 0$, $x = 8$ and $v = -15$.

(a) Show that $v = -\frac{2}{x}(x^2 - 4)$. [5]

(b) Find an expression for x in terms of t. [4]

Q18

9231/31, 9231/32 • May/June 2022 • Question 3

A particle P is moving in a horizontal straight line. Initially P is at the point O on the line and is moving with velocity $25\,\text{m s}^{-1}$. At time t s after passing through O, the acceleration of P is $\dfrac{4000}{(5t+4)^3}\,\text{m s}^{-2}$ in the direction PO. The displacement of P from O at time t is x m.

Find an expression for x in terms of t. [5]

Q19

9231/33 • May/June 2022 • Question 5

A particle P of mass 4kg is moving in a horizontal straight line. At time ts the velocity of P is $v\,\text{m s}^{-1}$ and the displacement of P from a fixed point O on the line is xm. The only force acting on P is a resistive force of magnitude $(4e^{-x}+12)\,e^{-x}$ N. When $t=0$, $x=0$ and $v=4$.

(a) Show by integration that $v=\dfrac{1+3e^{x}}{e^{x}}$. [4]

(b) Find an expression for x in terms of t. [4]

Q20

9231/31, 9231/33 • October/November 2022 • Question 4

A particle of mass 0.5 kg moves along a horizontal straight line. Its velocity is $v\,\mathrm{m\,s^{-1}}$ at time t s. The forces acting on the particle are a driving force of magnitude 50 N and a resistance of magnitude $2v^2$ N. The initial velocity of the particle is $3\,\mathrm{m\,s^{-1}}$.

(a) Find an expression for v in terms of t. [7]

(b) Deduce the limiting value of v. [1]

Q21

9231/32 • October/November 2022 • Question 4

A particle P of mass 5 kg moves along a horizontal straight line. At time t s, the velocity of P is $v\,\mathrm{m\,s^{-1}}$ and its displacement from a fixed point O on the line is x m. The forces acting on P are a force of magnitude $\frac{500}{v}$ N in the direction OP and a resistive force of magnitude $\frac{1}{2}v^2$ N. When $t = 0$, $x = 0$ and $v = 5$.

(a) Find an expression for v in terms of x. [6]

(b) State the value that the speed approaches for large values of x. [1]

Q22

9231/31, 9231/32 • May/June 2023 • Question 6

A particle P moving in a straight line has displacement xm from a fixed point O on the line and velocity v m s^{-1} at time ts. The acceleration of P, in m s^{-2}, is given by $6v\sqrt{v+9}$. When $t = 0$, $x = 2$ and $v = 72$.

(a) Find an expression for v in terms of x. [4]

(b) Find an expression for x in terms of t. [5]

Q23

9231/33 • May/June 2023 • Question 6

A particle of mass $m$kg falls vertically under gravity, from rest. At time ts, P has fallen xm and has velocity $v\,\text{m s}^{-1}$. The only forces acting on P are its weight and a resistance of magnitude $kmgv$ N, where k is a constant.

(a) Find an expression for v in terms of t, g and k. [5]

(b) Given that $k = 0.05$, find, in metres, how far P has fallen when its speed is $12\,\text{m s}^{-1}$. [5]

Q24

9231/31, 9231/33 • October/November 2023 • Question 2

A ball of mass 2 kg is projected vertically downwards with speed $5\,\mathrm{m\,s^{-1}}$ through a liquid. At time t s after projection, the velocity of the ball is $v\,\mathrm{m\,s^{-1}}$ and its displacement from its starting point is x m. The forces acting on the ball are its weight and a resistive force of magnitude $0.2v^2$ N.

(a) Find an expression for v in terms of t. [6]

(b) Deduce what happens to v for large values of t. [1]

Q25

9231/32 • October/November 2023 • Question 2

A particle P of mass 0.5 kg moves in a straight line. At time t s the velocity of P is v m s^{-1} and its displacement from a fixed point O on the line is x m. The only forces acting on P are a force of magnitude $\dfrac{150}{(x+1)^2}$ N in the direction of increasing displacement and a resistive force of magnitude $\dfrac{450}{(x+1)^3}$ N. When $t = 0$, $x = 0$ and $v = 20$.

Find v in terms of x, giving your answer in the form $v = \dfrac{Ax+B}{(x+1)}$, where A and B are constants to be determined. [6]

Q26

9231/31, 9231/32 • May/June 2024 • Question 6

A particle P of mass 2kg moving on a horizontal straight line has displacement xm from a fixed point O on the line and velocity v m s^{-1} at time t s. The only horizontal force acting on P has magnitude $\frac{1}{10}(2v-1)^2\mathrm{e}^{-t}$ N and acts towards O. When $t=0$, $x=1$ and $v=3$.

(a) Find an expression for v in terms of t. [5]

(b) Find an expression for x in terms of t. [4]

Q27

9231/33 • May/June 2024 • Question 7

A parachutist of mass $m$kg opens his parachute when he is moving vertically downwards with a speed of $50\,\text{m s}^{-1}$. At time t s after opening his parachute, he has fallen a distance xm from the point where he opened his parachute, and his speed is $v\,\text{m s}^{-1}$. The forces acting on him are his weight and a resistive force of magnitude mv N.

(a) Find an expression for v in terms of t. [6]

(b) Find an expression for x in terms of t. [3]

(c) Find the distance that the parachutist has fallen, since opening his parachute, when his speed is $15\,\text{m s}^{-1}$. [2]

Q28

9231/31, 9231/33 • October/November 2024 • Question 5

A particle P of mass 2 kg moving on a horizontal straight line has displacement x m from a fixed point O on the line and velocity $v\,\mathrm{m\,s^{-1}}$ at time t s. The only horizontal force acting on P is a variable force F N which can be expressed as a function of t. It is given that

$$\frac{v}{x} = \frac{3-t}{1+t}$$

and when $t = 0$, $x = 5$.

(a) Find an expression for x in terms of t. [4]

(b) Find the magnitude of F when $t = 3$. [3]

Q29

9231/32 • October/November 2024 • Question 7

A particle P of mass m kg is held at rest at a point O and released so that it moves vertically under gravity against a resistive force of magnitude $0.1mv^2$ N, where v m s^{-1} is the velocity of P at time t s.

(a) Find an expression for v in terms of t. [6]

The displacement of P from O at time t s is x m.

(b) Find an expression for v^2 in terms of x. [5]

Q30

9231/31, 9231/32 • May/June 2025 • Question 1

A particle P of mass 8 kg is moving in a straight horizontal line. At time t s, P has displacement x m from a fixed point O on the line and velocity $v\,\mathrm{m\,s^{-1}}$. The only horizontal force acting on P has magnitude $(x^3 + 4x)$ N and acts in the direction OP. When $t = 0$, $x = 0$ and $v = 1$.

(a) Find an expression for v in terms of x, giving your answer in the form $v = ax^2 + b$, where a and b are constants to be determined. [3]

(b) Find an expression for x in terms of t. [2]

Q31

9231/33 • May/June 2025 • Question 3

A ball of mass m kg is projected vertically upwards with initial speed $U\,\mathrm{m\,s^{-1}}$ and moves under gravity. At time t s after projection, the ball has travelled a distance x m and its speed is $v\,\mathrm{m\,s^{-1}}$. There is a resistive force of magnitude mkv^2 N, where k is a positive constant.

(a) Show that the distance travelled by the ball when it is moving upwards is $x = \dfrac{1}{2k}\ln\left(\dfrac{g+kU^2}{g+kv^2}\right)$. [4]

It is given that $k = 0.025$ and that $U = 20$.

(b) Find the time taken for the ball to reach its maximum height. [4]

Q32

9231/34 • May/June 2025 • Question 2

A particle P of mass m kg moves along a horizontal straight line against a resistive force of magnitude $2mv^3$ N, where $v\,\mathrm{m\,s^{-1}}$ is the velocity of P at time t s. When $t = 0$, $v = 1$.

(a) Find an expression for v in terms of t. [4]

(b) Find the displacement of P from its initial position when $t = 6$. [3]

Q33

9231/31, 9231/33 • October/November 2025 • Question 3

A particle P is moving in a straight horizontal line. At time t s, the displacement of P from a fixed point O on the line is x m and the velocity of P is $v\,\mathrm{m\,s^{-1}}$. The acceleration of P is $\frac{1}{2}(v^2+4)\,\mathrm{m\,s^{-2}}$ in the direction PO. Initially P is at O and is moving with velocity $2\,\mathrm{m\,s^{-1}}$.

(a) Find an expression for x in terms of t. [5]

(b) Find the time when P next goes through O. [2]

Q34

9231/34 • October/November 2025 • Question 5

A particle P of mass m kg is projected vertically upwards from a point O with an initial speed of $20\,\mathrm{m\,s^{-1}}$ and moves under gravity. There is a resistive force of magnitude $0.025mv^2$ N, where $v\,\mathrm{m\,s^{-1}}$ is the speed of P at time t s after projection. The displacement of P from O is x m at time t s after projection.

(a) Find an expression for v in terms of x, while P is moving upwards. [5]

(b) Find an expression for v in terms of t, while P is moving upwards. [4]

Q35

9231/32 • October/November 2025 • Question 7

A particle P of mass m kg moving along a rough horizontal table has displacement x m from a fixed point O on the table and velocity $v\,\mathrm{m\,s^{-1}}$ at time t s. The particle P is subject to a resistive force of magnitude $mgkv$ N, where k is a positive constant, and a frictional force of magnitude μmg. The particle P is initially at O with speed $U\,\mathrm{m\,s^{-1}}$.

(a) Show that $t = \dfrac{1}{gk}\ln\left(\dfrac{kU+\mu}{kv+\mu}\right)$. [4]

It is given that $U = 10$, $k = 0.04$ and $\mu = 0.2$.

(b) Find the distance P moves before coming to rest. [4]

(c) Find the average speed of P over the period it is moving. [2]

Written by Ali Hashir.

Answers available in the companion mark scheme booklet.