Further Math with Ali Hashir

Linear Motion Under a Variable Force

Topical Mark Schemes (CIE 9231)

Mark schemes corresponding to the topical questions booklet for this topic, compiled from official CIE 9231 marking schemes (2015–2025). Question numbers match the questions booklet.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Contents

Click a question number below to jump straight to its mark scheme. Every box is a live link.

For the best hyperlink experience, open this PDF in a dedicated reader – such as Adobe Acrobat, Apple Preview, or your browser's built-in PDF viewer – rather than a basic in-app or embedded preview, where links sometimes don't work.

Q1

9231/21, 9231/22 • May/June 2015 • Question 2

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Mark scheme.

(i)

Find ω^2 from SHM eqn $d^2x/dt^2 = -\omega^2 x$ at C: $0.625 = 10\omega^2$, $\omega^2 = 0.0625$ or $1/16$.	**B1**
Find period T from $T = 2\pi/\omega$ (ft on ω^2): $T = 2\pi/\frac{1}{4} = 8\pi$ (not 25.1).	**B1**

(ii)

Find amplitude a from $v_C^2 = \omega^2(a^2 - 10^2)$: $6^2 = \omega^2(a^2 - 10^2)$, $a^2 = 6^2 \times 16 + 10^2$, $a = \sqrt{676} = 26$.	**M1 A1**
Find time from C to M, e.g. $\omega^{-1}\sin^{-1}(10/a) + \omega^{-1}\sin^{-1}\frac{1}{2}$ (AEF throughout).	**M1**
$= \omega^{-1}\{0.3948 + \pi/6\ [= 0.5236]\}$ (or equivalent forms).	**A1**
$= 1.579 + 2.094 = 3.67$ s.	**A1; A1**
	Total: 8

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Q2

9231/23 • May/June 2015 • Question 3

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Mark scheme.

State or imply position where maximum speed is next attained: centre O.	**M1**
Find ω from SHM eqn $x = a\sin\omega t$, e.g. $1 = \omega^{-1}\sin^{-1}\frac{1}{2}$, $\omega = \pi/6$.	**M1 A1**
Find period T from $T = 2\pi/\omega$ (ft on ω): $T = 2\pi/(\pi/6) = 12$ s.	**B1**
Find distance x from O when speed is half maximum: $\omega^2(a^2 - x^2) = \frac{1}{4}v_{\max}^2$.	**M1**
Use maximum speed $v_{\max} = a\omega$ (note x is independent of ω).	**M1**
$x^2 = \frac{3}{4}a^2$, $x = \sqrt{3}a/2$ or $0.866a$.	**A1**
Find a from $x = a\sin\omega t$: $2.5 = \omega^{-1}\sin^{-1}(\sqrt{2}/a) - \omega^{-1}\sin^{-1}(-\frac{1}{2})$, giving $\sin^{-1}(\sqrt{2}/a) = \pi/4$, $a = 2$ m.	**M1 A1**
	Total: 9

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Q3

9231/21, 9231/22 • May/June 2016 • Question 3

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Mark scheme.

EITHER: Find time during which speed is low, e.g. $x = a\cos\omega t$, $v = (\pm)a\omega\sin\omega t$	**M1**
$\frac{1}{2}a\omega = a\omega\sin\omega t$ (working may be implied): $t_L = \pi/6\omega$ or $T/12$	**M1 A1**
Find ω (implied) or T from given time: $4t_L = 4/3$, $\omega = \pi/2$ or $T = 4$ s	**M1 A1**
Find other value: $T = 2\pi/\omega = 4$ s or $\omega = \pi/2$	**B1**
Find maximum speed $v_{\max}$: $v_{\max} = 0.25\omega = \pi/8$ or 0.393 m s^{-1}	**M1 A1**
	Total: 8

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Q4

9231/21, 9231/22, 9231/23 • October/November 2016 • Question 1

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Mark scheme.

(i)	
Find ω^2 from $v^2 = \omega^2(a^2 - x^2)$: $4^2 = \omega^2(3^2 - 1^2)$, $\omega^2 = 2$	**M1**
$\omega^2 = 2$	**A1**
Find max. acceleration from $\mathrm{d}^2x/\mathrm{d}t^2 = -\omega^2 x$, $\sqrt{}$ on ω^2: $2 \times 3 = 6$ [m s^{-2}] (allow -6) (FT on their ω^2)	**A1**
(ii)	
Find number of oscillations in 60 s from $T = 2\pi/\omega$: $60/(2\pi/\omega)$ [$= 60/4.443 = 13.5$]	**M1**
13 (allow M1 A0 for $60/(\pi/\omega)$ [$= 27$])	**A1**
(iii)	
Find time from A to C, e.g. $\omega^{-1}\sin^{-1}(1) + \omega^{-1}\sin^{-1}\frac{1}{3}$ or $\frac{1}{4}T + \omega^{-1}\sin^{-1}\frac{1}{3}$ [$= 1.111 + 0.240$] or $\omega^{-1}\cos^{-1}(-\frac{1}{3})$ or $\frac{1}{2}T - \omega^{-1}\cos^{-1}\frac{1}{3}$ [$= 2.221 - 0.870$]	**M1**
$= 1.91/\omega$	**A1**
$= 1.35$ s	**A1**
	Total: 8

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Q5

9231/21, 9231/22, 9231/23 • October/November 2017 • Question 2

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Mark scheme.

Use $v^2 = \omega^2(a^2 - x^2)$ in each position: $\left(\frac{3\pi}{5}\right)^2 = \omega^2\left(a^2 - (a-1.6)^2\right)$.	**M1, A1**
Also $\left(\frac{\pi}{4}\right)^2 = \omega^2\left(a^2 - (a-0.2)^2\right)$; combine: $\left(\frac{3\pi}{5}\right)^2(0.4a - 0.04) = \left(\frac{\pi}{4}\right)^2(3.2a - 2.56)$.	**A1**
$a = 2.6$ m.	**M1, A1**
$\omega^2 = \left(\frac{\pi}{4}\right)^2$ or $\omega = \frac{\pi}{4}$.	**A1**
$T = 2\pi/(\pi/4) = 8$ s.	**B1FT**
	Total: 7

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Q6

9231/21, 9231/23 • October/November 2018 • Question 1

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Mark scheme.

$\omega = 2\pi/T = 4$ (find ω from period T, may be implied)	**B1**
$v = \omega\sqrt{a^2 - x^2}$	**M1**
$v = 4\sqrt{3^2 - 1^2} = 8\sqrt{2}$ or 11.3 [m s^{-1}]	**A1**
	Total: 3

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Q7

9231/21, 9231/23 • October/November 2018 • Question 5

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Mark scheme.

(i)	
$\lambda(AE - 1.25)/1.25 =$ – verify AE by equating equilibrium tensions	**M1A1**
$= 0.6\lambda(2.6 - AE - 1.0)/1.0$ (AEF) (SC: vertical motion can earn M1 only)	**A1**
$AE - 1.25 = 1.2 - 0.75AE$, $AE = 1.4$ (AG)	**A1**
(ii)	
$\pm m\,\mathrm{d}^2x/\mathrm{d}t^2 = -\lambda(0.15 + x)/1.25 + 0.6\lambda(0.2 - x)/1.0$ or equivalent – apply Newton's law at $1.4 + x$ or $1.4 - x$ from A (lose A1 for one incorrect tension term)	**M1A2**
Simplify to give SHM equation in standard form: $\mathrm{d}^2x/\mathrm{d}t^2 = -(1.4\lambda/0.4)x = -3.5\lambda x$ (A0 if no minus sign, or direction of acceleration undefined; SC: B2 if result stated without derivation, max 2/5)	**M1A1**
(iii)	
$\omega = 2\pi/(\pi/7) = 14 = \sqrt{3.5\lambda}$ (find equation for λ using $T = 2\pi/\omega$ with FT on ω from SHM equation)	**M1A1**
$\lambda = 14^2/3.5 = 56$	**A1**
	Total: 12

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Q8

9231/22 • October/November 2018 • Question 1

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Mark scheme.

$v_A^2 = \omega^2(a^2 - 0.1^2)$ and $v_B^2 = \omega^2(a^2 - 0.5^2)$ (use $v^2 = \omega^2(a^2 - x^2)$ at A and B, may be implied)	**B1**
$a^2 - 0.1^2 = 2(a^2 - 0.5^2)$ (find amplitude a from ratio 2 of $[\frac{1}{2}m]v_A^2$ to $[\frac{1}{2}m]v_B^2$)	**M1**
$a^2 = 0.5 - 0.01 = 0.49$, $a = 0.7$ [m] (taking ratio $\frac{1}{2}$ loses A1)	**A1**
	Total: 3

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Q9

9231/21, 9231/22 • May/June 2019 • Question 2

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Mark scheme.

Find amplitude a (allow M1 for $v_A = 2v_B$): $v_B = 2v_A$, $v_B^2 = 4v_A^2$, $\omega^2(a^2 - 1^2) = 4\omega^2(a^2 - 3.5^2)$	**M1**
$3a^2 = 48$, $a = [\pm]4\,\text{m}$	**A1**
Find ω from given maximum acceleration: $1 = a\omega^2$, $\omega = 1/\sqrt{a}$ $[= 1/2]$	**B1**
Find speed v_O at O: $v_O = a\omega = \sqrt{a} = 2\,\text{m}\,\text{s}^{-1}$ (ii)	**B1**
Find equation (AEF) for ωt_{AB}, combining t_A and t_B, using $x = a\sin\omega t$: $\omega t_{AB} = \sin^{-1}(3.5/a) + \sin^{-1}(1/a) = \sin^{-1}0.875 + \sin^{-1}0.25$	**M1 A1**
$= 1.065 + 0.253$ (or $t_{AB} = 2.131 + 0.505$)	**A1**
Hence find t_{AB}: $t_{AB} = 2 \times 1.318 = 2.64\,\text{s}$	**A1**
	Total: 8

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Q10

9231/31, 9231/32 • May/June 2020 • Question 5

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Mark scheme.

$\dfrac{dv}{3u-v} = k\,dt.$	**M1**
$-\ln(3u-v) = kt + d;\ t = 0, v = u : d = -\ln 2u.$	**M1**
$v = 2u : t = \dfrac{1}{k}\ln 2.$ **(b)**	**A1**
$v\dfrac{dv}{dx} = 3ku - kv \left[\Rightarrow \dfrac{v\,dv}{3u-v} = k\,dx\right].$	**B1**
$\dfrac{(-(3u-v)+3u)\,dv}{3u-v} = k\,dx$ so $-v - 3u\ln(3u-v) = kx + c.$	**M1A1**
$x = 0, v = u : c = -u - 3u\ln 2u.$	**M1**
$v = 2u : x = \dfrac{u}{k}(3\ln 2 - 1).$	**A1**
	Total: 8

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Q11

9231/33 • May/June 2020 • Question 2

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Mark scheme.

$\dfrac{dv}{g-kv} = dt.$	**M1**
$-\dfrac{1}{k}\ln(g-kv) = t + A.$	**M1**
$t=0, v=0 : A = -\dfrac{1}{k}\ln g.$	**A1**
$t = -\dfrac{1}{k}\ln\dfrac{g-kv}{g}.$	**M1**
$v = \dfrac{g}{k}\left(1-e^{-kt}\right).$	**M1A1**
	Total: 6

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Q12

9231/31, 9231/33 • October/November 2020 • Question 7

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Mark scheme.

Correct equation and attempt to integrate: $v\dfrac{dv}{dx} = -\dfrac{100}{x^3} + \dfrac{200}{x^2}$; $\dfrac{v^2}{2} = \dfrac{50}{x^2} - \dfrac{200}{x} + A$	**M1A1**
Use initial condition $x = 1, v = -10$: $A = 200$	**M1**
Rearrange to find v^2: $v^2 = \dfrac{100(2x-1)^2}{x^2}$	**M1**
$v = \pm\dfrac{10(2x-1)}{x}$, take negative sign to meet initial condition, so $v = \dfrac{10(1-2x)}{x}$ (AG; convincingly shown, no mention of $\pm$ scores A0) (b)	**A1**
Rearrange and attempt to integrate: $\dfrac{x\,dx}{1-2x} = 10\,dt$; $\dfrac{1}{2}\left(\dfrac{1}{1-2x} - 1\right)dx = 10\,dt$; $-\dfrac{1}{4}\ln\lvert 1-2x\rvert - \dfrac{x}{2} = 10t + B$	**M1A1**
Use initial condition $t = 0, x = 1$: $B = -\dfrac{1}{2}$	**M1**
$2x - 2 = -40t - \ln\lvert 1-2x\rvert$ so $e^{-40t} = (2x-1)e^{2x-2}$ (AG; convincingly shown, working required)	**A1**
For large values of t, $x \to \dfrac{1}{2}$ (CAO)	**B1**
	Total: 10

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Q13

9231/32 • October/November 2020 • Question 7

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Mark scheme.

N2L: $mv\dfrac{dv}{dx} = -mkv^2$	**B1**
Separate variables and integrate: $\ln v = -kx + c$	**M1**
$x = 0, v = u$: $c = \ln u$	**M1**
$v = \frac{1}{2}u$: $\ln\frac{1}{2} = -kx$, so $x = \dfrac{1}{k}\ln 2$ (AG) (b)	**A1**
N2L (allow missing m): $mv\dfrac{dv}{dx} = -mkv^2 + \dfrac{5m}{v}$	**B1**
Separate variables and integrate: $\dfrac{v^2\,dv}{5 - kv^3} = dx$; $-\dfrac{1}{3k}\ln(5 - kv^3) = x(+d)$	**M1A1**
Use condition from (a): $-\dfrac{1}{3k}\ln(5 - kv^3) = x - \dfrac{1}{3k}\ln(5 - ku^3) - \dfrac{1}{k}\ln 2$ (M0 if $v = \frac{1}{2}u, x = 0$ used unless $\frac{1}{k}\ln 2$ added later; rearrange dependent on ln solution)	**M1M1**
Use $v = u$: $x = \dfrac{1}{3k}\ln\left(\dfrac{40 - ku^3}{5 - ku^3}\right)$	**M1A1**
	Total: 11

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Q14

9231/31, 9231/32 • May/June 2021 • Question 1

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Mark scheme.

Separate variables: $\dfrac{dv}{\sqrt{v}} = -\dfrac{10\,dt}{(t+1)^2}$.	**M1**
Attempt to integrate: $2\sqrt{v} = \dfrac{10}{t+1} + A$.	**M1 A1**
Use correct initial condition: $t = 0, v = 25, A = 0$.	**M1**
$v = \dfrac{25}{(t+1)^2}$ (CAO).	**A1**
	Total: 5

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Q15

9231/33 • May/June 2021 • Question 5

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Mark scheme.

$m\dfrac{dv}{dt} = -mg - 2mv$ (N2L, must include m; use of SUVAT implies 0 marks).	**B1**
Separate variables and integrate 3-term N2L: $\ln(5+v) = -2t(+A)$.	**M1**
$\ln(5+v) = -2t + A$ (FT only sign error in N2L).	**A1 FT**
Use correct initial condition: $t = 0, v = 20, A = \ln 25$.	**M1**
Remove all logs: $2t = \ln\left(\dfrac{25}{5+v}\right)$, $e^{2t} = \dfrac{25}{5+v}$.	**M1**
$v = 25e^{-2t} - 5$. (b)	**A1**
$x = -\dfrac{25}{2}e^{-2t} - 5t(+B)$ (integrate expression from part (a); use of SUVAT implies 0 marks).	**M1**
$t = 0, x = 0, B = \frac{25}{2}$, so $x = \frac{25}{2}(1 - e^{-2t}) - 5t$. (c)	**A1 FT**
Greatest height when $v = 0$, so $t = 0.8047\ldots$ or $\frac{1}{2}\ln 5$ (use of SUVAT in part (a) or (b) implies 0 marks).	**M1**
$x = 5.98$ m (CWO).	**A1**
	Total: 10

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Q16

9231/31, 9231/33 • October/November 2021 • Question 2

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Mark scheme.

(a)

Separate variables and integrate: $\frac{dv}{v} = \left(\frac{1-2t^2}{t}\right) dt$ so $\ln\|v\| = \ln t - t^2 + c$ (must include logs, condone missing modulus).	**M1 A1**
$\|v\| = Ate^{-t^2}$, $-v = Ate^{-t^2}$, $v = -Ate^{-t^2}$ (CAO).	**A1**

(b)

Substituting their answer to part (a) into given formula: $a = \frac{-Ate^{-t^2}(1-2t^2)}{t} = -Ae^{-t^2}(1-2t^2)$.	**M1**
Use initial condition: $t = 1,\ a = 5\ (A = 5e)$.	**M1**
Use N2L, correct work only: Force $= 5me^{1-t^2}(2t^2-1)$.	**A1**
	Total: 6

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Q17

9231/32 • October/November 2021 • Question 6

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Mark scheme.

(a)	
Separate variables and integrate, $+c$ not required for M1: $2v\dfrac{dv}{dx} = 8x - \dfrac{128}{x^3}$.	**M1**
$v^2 = 4x^2 + 64x^{-2} + c$ (OE).	**A1**
Use initial condition: $x = 8$, $v = -15$ and $c = -32$.	**M1**
Correct expression for v^2, AEF: $v^2 = \dfrac{4}{x^2}(x^4 - 8x^2 + 16)$ or $4x^2 - 32 + \dfrac{64}{x^2}$.	**A1**
$v^2 = \dfrac{4}{x^2}(x^2-4)^2$ giving $v = -\dfrac{2}{x}(x^2-4)$ (convincingly shown, e.g. v is negative initially, AG).	**A1**
(b)	
Use $v = \dfrac{dx}{dt}$ and integrate: $\dfrac{1}{2}\ln(x^2-4) = -2t(+A)$.	**M1**
Use initial condition: $t = 0$, $x = 8$, $A = \dfrac{1}{2}\ln 60$.	**DM1**
Remove log: $\dfrac{1}{2}\ln\left(\dfrac{x^2-4}{60}\right) = -2t$ giving $\dfrac{x^2-4}{60} = e^{-4t}$.	**M1**
$x = \sqrt{4 + 60e^{-4t}}$ (CAO).	**A1**
	Total: 9

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Q18

9231/31, 9231/32 • May/June 2022 • Question 3

↑ Back to contents

Mark scheme.

Integrate: $\frac{dv}{dt} = -\frac{4000}{(5t+4)^3}$; $v = \frac{400}{(5t+4)^2}(+A)$ (constant of integration needed for A1).	**M1 A1**
Find constant: $t = 0,\ v = 25 \Rightarrow A = 25 - 25 = 0$.	**M1**
Integrate and find constant: $v = \frac{dx}{dt}$: $x = -80(5t+4)^{-1} + B$; $x = 0,\ t = 0 \Rightarrow$ $B = 20$.	**M1**
$x = \frac{-80}{5t+4} + 20 \left(= \frac{100t}{5t+4}\right)$.	**A1**
	Total: 5

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Q19

9231/33 • May/June 2022 • Question 5

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Mark scheme.

$4v\dfrac{dv}{dx} = -(4e^{-x}+12)\,e^{-x}$.	**B1**
Expression of the correct form: $\dfrac{1}{2}v^2 = \dfrac{1}{2}e^{-2x} + 3e^{-x}(+A)$.	**M1**
$v = 4,\ x = 0 \Rightarrow A = \dfrac{9}{2}$.	**A1**
$v^2 = e^{-2x} + 6e^{-x} + 9 = (3+e^{-x})^2 \Rightarrow v = 3 + e^{-x} = \dfrac{1+3e^x}{e^x}$ (AG; must see the factorisation, condone lack of justification for taking positive square root). (b)	**A1**
$\dfrac{dx}{dt} = \dfrac{1+3e^x}{e^x}$ so $\displaystyle\int \frac{e^x}{3e^x+1}dx = \int 1\,dt$; $\dfrac{1}{3}\ln(3e^x+1) = t(+B)$ (integration to obtain ln term; correct answer with constant of integration).	**M1* A1**
Find the constant and substitute into their general solution: $t = 0,\ x = 0 \Rightarrow B = \dfrac{1}{3}\ln 4$; $3t = \ln\dfrac{3e^x+1}{4}$.	**DM1**
$x = \ln\left(\dfrac{4}{3}e^{3t} - \dfrac{1}{3}\right)$ (OE).	**A1**
	Total: 8

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Q20

9231/31, 9231/33 • October/November 2022 • Question 4

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Mark scheme.

N2L: $0.5\dfrac{dv}{dt} = 50 - 2v^2$, i.e. $\dfrac{dv}{dt} = 4(25 - v^2)$	**B1**
Separate variables and use partial fractions: $\dfrac{1}{10}\int\left(\dfrac{1}{5-v} + \dfrac{1}{5+v}\right)dv = \int 4\,dt$	**M1**
Integrate into log terms: $\dfrac{1}{10}\left(-\ln(5-v) + \ln(5+v)\right) = 4t + A$	**M1,A1**
Use initial condition $t = 0, v = 3$: $A = \frac{1}{10}\ln 4$	**M1**
Rearrange to make v the subject: $4t = \frac{1}{10}\ln\dfrac{5+v}{4(5-v)}$ leading to $\dfrac{5+v}{20-4v} = e^{40t}$	**M1**
$v = \dfrac{5(4 - e^{-40t})}{4 + e^{-40t}}$ **(b)**	**A1**
As $t \to \infty$, $v \to 5$.	**B1**
	Total: 8

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Q21

9231/32 • October/November 2022 • Question 4

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Mark scheme.

Sight of m or 5 required: $5v\dfrac{dv}{dx} = \dfrac{500}{v} - \frac{1}{2}v^2$	**B1**
Separate variables and attempt to integrate into a log term: $\dfrac{10v^2\,dv}{1000 - v^3} = dx$	**M1**
$-\dfrac{10}{3}\ln(1000 - v^3) = x(+A)$	**A1**
Evaluate constant using $x = 0, v = 5$: $A = -\dfrac{10}{3}\ln 875$	**M1**
Make v the subject: $x = \dfrac{10}{3}\ln\dfrac{875}{1000 - v^3}$	**M1**
$v = \left[1000 - 875e^{-0.3x}\right]^{1/3}$ **(b)**	**A1**
Maximum value of v is 10 (no FT: found from initial equation).	**B1**
	Total: 7

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Q22

9231/31, 9231/32 • May/June 2023 • Question 6

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Mark scheme.

$v\dfrac{dv}{dx} = 6v\sqrt{v+9}$ and attempt to separate variables and integrate.	**M1**
$2\sqrt{v+9} = 6x + A$.	**A1**
Use initial condition to find constant: $x = 2, v = 72 \Rightarrow A = 6$.	**M1**
$v = 9(x+1)^2 - 9$ (correct, AEF). (b)	**A1**
Separate variables and write in the form $\left(\dfrac{a}{x} - \dfrac{b}{x+c}\right) dx = dt$: $\dfrac{dx}{dt} = 9(x^2 + 2x)$, $\dfrac{dx}{x(x+2)} = 9dt$, $\dfrac{1}{2}\left(\dfrac{1}{x} - \dfrac{1}{x+2}\right) dx = 9dt$.	**M1**
Integrate, any correct form: $\dfrac{1}{2}\ln\left(\dfrac{x}{x+2}\right) = 9t + B$.	**A1**
Use initial condition to find constant: $t = 0, x = 2 \Rightarrow B = \dfrac{1}{2}\ln\dfrac{1}{2}$.	**M1**
Take logarithms: $18t = \ln\left(\dfrac{2x}{x+2}\right)$, $e^{18t} = \dfrac{2x}{x+2}$.	**M1**
$x = \dfrac{2e^{18t}}{2 - e^{18t}}$ or $x = \dfrac{2}{2e^{-18t} - 1}$ (any correct form).	**A1**
	Total: 9

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Q23

9231/33 • May/June 2023 • Question 6

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Mark scheme.

$m\dfrac{dv}{dt} = mg(1 - kv)$. Mass must be seen at this point or earlier. [SUVAT does not apply.]	**B1**
Separate variables and integrate to logarithm: $-\dfrac{1}{k}\ln(1 - kv) = gt + A$.	**M1**
Correct, with constant of integration.	**A1**
Use initial condition to evaluate their constant: $t = 0, v = 0 \Rightarrow A = 0$.	**M1**
$v = \dfrac{1}{k}\left(1 - e^{-kgt}\right)$ (any correct form with v as subject; final A0 if numerical value of g present). (b)	**A1**
$k = 0.05$ and so $\dfrac{dx}{dt} = 20(1 - e^{-0.5t})$ (attempt to integrate if expression contains a term of the form be^{ct}).	**M1**
Integrate: $x = 20\left(t + 2e^{-0.5t}\right) + B$.	**A1**
$t = 0, x = 0 \Rightarrow B = -40$ (use initial condition to evaluate their constant).	**DM1**
When $v = 12$, from part (a), $e^{-0.5t} = 1 - 0.05 \times 12 = 0.4$, $t = -2\ln 0.4$.	**M1**
$x = -40\ln 0.4 + 40 \times 0.4 - 40 = 12.7$ [$= 40\ln\dfrac{5}{2} - 24$].	**A1**
	Total: 10

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Q24

9231/31, 9231/33 • October/November 2023 • Question 2

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Mark scheme.

$2\dfrac{dv}{dt} = 2g - 0.2v^2$	B1	**(a)**
Separate variables and attempt to integrate $\dfrac{dv}{0.1(100-v^2)} = dt$	M1	
$\dfrac{1}{20}\ln\left(\dfrac{10+v}{10-v}\right) = 0.1t + c$	A1	
$t = 0,\ v = 5$: $c = \frac{1}{20}\ln 3$	M1	
$2t = \ln\dfrac{10+v}{3(10-v)},\ e^{2t} = \dfrac{10+v}{3(10-v)}$	M1	
$v = \dfrac{30 - 10e^{-2t}}{3 + e^{-2t}}$	A1	
$v \to 10$	B1FT	**(b)**
		Total: 7

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Q25

9231/32 • October/November 2023 • Question 2

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Mark scheme.

$0.5v\dfrac{dv}{dx} = \dfrac{150}{(x+1)^2} - \dfrac{450}{(x+1)^3}$	M1
Integrate: $0.5v^2 = -\dfrac{300}{x+1} + \dfrac{450}{(x+1)^2} + A$	M1A1
$x = 0$, $v = 20$; $A = 50$	M1
Rearrange: $v^2 = \dfrac{100(x^2 - 4x + 4)}{(x+1)^2}$	A1
$v^2 = \dfrac{100(x-2)^2}{(x+1)^2}$, so $v = \pm\dfrac{10(x-2)}{(x+1)}$; from initial condition, sign must be negative, $v = \dfrac{20-10x}{x+1}$	A1
	Total: 6

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Q26

9231/31, 9231/32 • May/June 2024 • Question 6

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Mark scheme.

*M1: $2\dfrac{\mathrm{d}v}{\mathrm{d}t} = -\dfrac{1}{10}(2v-1)^2\mathrm{e}^{-t}$ so $\dfrac{\mathrm{d}v}{(2v-1)^2} = -\dfrac{1}{20}\mathrm{e}^{-t}\,\mathrm{d}t$ (separate variables and attempt to integrate both sides; where p and q are constants, $\dfrac{p}{(2v-1)} = q\mathrm{e}^{-t} + A$).	**(a)**
$-\dfrac{1}{2(2v-1)} = +\dfrac{1}{20}\mathrm{e}^{-t} + A$ (AEF).	**A1**
$t = 0, v = 3$: $A = -\dfrac{3}{20}$ (substituting the boundary condition and obtain a value).	**DM1**
$v = \dfrac{1}{2} + \dfrac{5\mathrm{e}^t}{3\mathrm{e}^t - 1}$ (find v in terms of t, AEF).	***M1 A1**
*M1: Integrate: $x = pt + q\ln(r\mathrm{e}^{\pm t} - s)$ [$+B$].	**(b)**
$x = \dfrac{1}{2}t + \dfrac{5}{3}\ln(3\mathrm{e}^t - 1)$ [$+B$] (AEF).	**A1**
$t = 0, x = 1$: $B = 1 - \dfrac{5}{3}\ln 2$ (substituting the boundary condition and obtain a value).	**DM1**
$x = 1 + \dfrac{1}{2}t + \dfrac{5}{3}\ln\dfrac{(3\mathrm{e}^t - 1)}{2}$ (AEF).	**A1**
	Total: 9

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Q27

9231/33 • May/June 2024 • Question 7

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Mark scheme.

B1: $m\dfrac{\mathrm{d}v}{\mathrm{d}t} = m(10 - v)$ (no marks in this part if suvat used; must have sight of m, e.g. in $F = ma$).	**(a)**
$-\ln\lvert 10 - v\rvert = t + A$ or $-\ln(v - 10) = t + A$ (separate variables and integrate to obtain a ln term; constant may be omitted, constant needed for A1).	***M1 A1**
Use $t = 0, v = 50$: $A = -\ln\lvert -40\rvert$ (find constant, dependent on previous M1; may use limits instead).	**DM1**
$0.1ve^t = 4 + e^t$ (remove all logs).	**M1**
$v = 10 + 40e^{-t}$ (correct work only).	**A1**
M1: $x = 10t - 40e^{-t} + B$ (no marks in this part if suvat used in part (a) or (b); integrate their answer to part (a); constant may be omitted).	**(b)**
Use $t = 0, x = 0$: $B = 40$ (use initial condition in their expression for x in terms of t).	**M1**
$x = 10t - 40e^{-t} + 40$.	**A1**
M1: When $v = 15$, $e^{-t} = \dfrac{1}{8}$, $t = 2.08$ or $\ln 8$ (no marks in this part if suvat used in part (a), (b) or (c); find value of t from their answer to part (a)).	**(c)**
$x = 55.8$ (metres) (note $35 + 10\ln 8$ scores A0).	**A1**
	Total: 11

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Q28

9231/31, 9231/33 • October/November 2024 • Question 5

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Mark scheme.

(a)	
Separate variables: $\dfrac{\mathrm{d}x}{x} = \left(\dfrac{4}{t+1} - 1\right)\mathrm{d}t$, obtain RHS in integrable form.	**M1**
$\ln\lvert x\rvert = 4\ln\lvert t+1\rvert - t + A$.	**A1**
Use $t = 0, x = 5$: $A = \ln 5$.	**M1**
$x = 5(t+1)^4\mathrm{e}^{-t}$.	**A1**
(b)	
$v = (3-t) \times 5(t+1)^3\mathrm{e}^{-t}$; Acceleration $= \dfrac{\mathrm{d}v}{\mathrm{d}t} =$ $5\mathrm{e}^{-t}\left(-(t+1)^3 + (3-t)3(t+1)^2 - (3-t)(t+1)^3\right) = 5\mathrm{e}^{-t}(t+1)^2(5-t)(1-t)$.	**M1**
$F = 2 \times$ acceleration, so at t: $F = 10\mathrm{e}^{-t}(t+1)^2(5-t)(1-t)$.	**M1**
At $t = 3$, magnitude of force is $640\mathrm{e}^{-3}\,\mathrm{N} \approx 31.9\,\mathrm{N}$.	**A1**
	Total: 7

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Q29

9231/32 • October/November 2024 • Question 7

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Mark scheme.

(a)

$m\dfrac{\mathrm{d}v}{\mathrm{d}t} = mg - 0.1mv^2$ (must see m).	**B1**
$\dfrac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.1v^2 = \dfrac{1}{10}(100 - v^2)$; separate variables and integrate: $\ln\left\|\dfrac{v+10}{10-v}\right\| = 2t + A$.	**M1***
Must see modulus sign.	**A1**
Use $t = 0, v = 0,\ A = 0$.	**DM1**
Remove logs to obtain v in terms of t.	**M1**
$v = \dfrac{10(\mathrm{e}^{2t} - 1)}{\mathrm{e}^{2t} + 1}$ (AEF).	**A1**

(b)

$v\dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{1}{10}(100 - v^2)$; separate variables and integrate: $-\frac{1}{2}\ln\|100 - v^2\| = \frac{1}{10}x + B$.	**M1***
Allow missing modulus sign.	**A1**
Use $x = 0, v = 0$: $B = -\frac{1}{2}\ln 100$.	**DM1**
Remove logs to obtain v^2 in terms of x.	**M1**
$v^2 = 100(1 - \mathrm{e}^{-x/5})$ (AEF).	**A1**
	Total: 11

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Q30

9231/31, 9231/32 • May/June 2025 • Question 1

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Mark scheme.

(a)

Correct use of N2L: $x^3 + 4x = 8v\dfrac{dv}{dx}$.	**B1**
Attempt to integrate by separating variables: $\int(x^3+4x)\,dx = \int 8v\,dv$, leading to $4v^2 = \frac{1}{4}x^4 + 2x^2 + c$. Must have $v\frac{dv}{dx}$; must include $+c$.	**M1**
Use initial conditions to find c and rearrange: $v^2 = \frac{1}{16}x^4 + \frac{1}{2}x^2 + 1 = \left(\frac{1}{4}x^2+1\right)^2$, so $v = \frac{1}{4}x^2 + 1$.	**A1**

(b)

Separate variables and integrate to $\tan^{-1}$ form: $\dfrac{dx}{dt} = \frac{1}{4}x^2 + 1$ leading to $\int \dfrac{1}{\frac{1}{4}x^2+1}dx = \int dt$, i.e. $t = 4\int \dfrac{1}{x^2+4}dx = 2\tan^{-1}\left(\frac{1}{2}x\right)$ $[+C]$. Only FT their v of form ax^2+b.	**M1**
$C = 0$ so $x = 2\tan\left(\frac{1}{2}t\right)$.	**A1**
	Total: 5

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Q31

9231/33 • May/June 2025 • Question 3

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Mark scheme.

(a)	
Correct use of N2L with m seen: $-mg - mkv^2 = mv\dfrac{dv}{dx}$.	**B1**
Attempt to integrate by separating variables: $\int dx = -\int \dfrac{v}{g+kv^2}dv$ leading to $x = -\dfrac{1}{2k}\ln(g+kv^2)[+c]$.	**M1**
$v = U$ when $x = 0$, so $c = \dfrac{1}{2k}\ln(g+kU^2)$.	**M1**
$x = \dfrac{1}{2k}\ln\left(\dfrac{g+kU^2}{g+kv^2}\right)$ (AG, shown convincingly).	**A1**
(b)	
Attempt to integrate by separating variables: $10 + \frac{1}{40}v^2 = -\dfrac{dv}{dt}$, i.e. $\int dt = -\int \dfrac{40}{400+v^2}dv$.	***M1**
Correct integration: $t = -\frac{40}{20}\tan^{-1}\left(\frac{1}{20}v\right)(+C)$.	**A1**
Find C using $t = 0, v = 20$ so $C = \frac{1}{2}\pi$, then substitute $v = 0$: $t = -2\tan^{-1}(0) + \frac{1}{2}\pi$.	**DM1**
$t = 1.57$ seconds (CWO, accept $\frac{1}{2}\pi$).	**A1**
	Total: 8

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Q32

9231/34 • May/June 2025 • Question 2

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Mark scheme.

(a)	
N2L (m must be present), attempt to integrate by separating variables: $m\frac{dv}{dt} = -2mv^3$, i.e. $\int v^{-3}dv = \int -2\,dt$.	**M1**
$-\frac{1}{2}v^{-2} = -2t + A$ (any correct equivalent form).	**A1**
Use $t = 0, v = 1$ to find $A = -\frac{1}{2}$.	**M1**
$v = \dfrac{1}{\sqrt{4t+1}}$ (AEF).	**A1**
(b)	
$\dfrac{dx}{dt} = \dfrac{1}{\sqrt{4t+1}}$ so $x = \frac{1}{2}\sqrt{4t+1}[+B]$ (FT their (a)).	**M1 FT**
Use $t = 0, x = 0$ to find $B = -\frac{1}{2}$.	**M1**
When $t = 6$, $x = 2$ metres (CAO).	**A1**
	Total: 7

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Q33

9231/31, 9231/33 • October/November 2025 • Question 3

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Mark scheme.

Forming differential equation and integrating to a term involving an inverse tangent: $\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{2}(v^2+4)$, $\frac{1}{2}\tan^{-1}(\frac{1}{2}v) = -\frac{1}{2}t\,[+A]$.	***M1**
CWO, condone omission of $+A$.	**A1**
Use initial condition to find constant: $t=0, v=2 : A = \frac{1}{2}\tan^{-1}(2/2) = \frac{1}{8}\pi$.	**DM1**
Integrate into term involving ln of cosine or sine: $v = \frac{\mathrm{d}x}{\mathrm{d}t} = 2\tan(\frac{1}{4}\pi - t)$, $x = 2\ln(\cos(t-\frac{1}{4}\pi))\,[+B]$.	**DM1**
Use initial condition and obtain correct expression: $x=0, t=0 : B = \ln 2$, so $x = 2\ln(\cos(t-\frac{1}{4}\pi)) + \ln 2$. **(b)**	**A1**
Solve equation to find a value for t: $2\ln(\cos(t-\frac{1}{4}\pi)) + \ln 2 = 0$.	**M1**
CWO: $t = \frac{1}{2}\pi$.	**A1**
	Total: 7

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Q34

9231/34 • October/November 2025 • Question 5

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Mark scheme.

m must be seen: $mv\dfrac{\mathrm{d}v}{\mathrm{d}x} = -0.025v^2m - 10m$.	**B1**
Separate variables and integrate to a term involving $\ln(av^2+b)$ or $\ln\left(\dfrac{av+b}{pv+q}\right)$: $\displaystyle\int \frac{v}{10+0.025v^2}\,\mathrm{d}v = -\int \mathrm{d}x$.	***M1**
Ignore modulus signs, condone missing $+c$: $-0.025x[+c] = \frac{1}{2}\ln(v^2+400)$.	**A1**
Use initial condition to find constant of integration: $c = \frac{1}{2}\ln 800$.	**DM1**
AEF: $v = 20\sqrt{2\mathrm{e}^{-\frac{1}{20}x} - 1}$. **(b)**	**A1**
Equation and integration to a term involving $\tan^{-1}$ (accept missing m): $m\dfrac{\mathrm{d}v}{\mathrm{d}t} = -0.025v^2m - 10m$.	**M1**
Condone missing $+k$: $-t[+k] = 2\tan^{-1}\dfrac{v}{20}$.	**A1**
Use initial condition with a term involving $\tan^{-1}(v/a)$: $k = 2\tan^{-1}\dfrac{20}{20} = \frac{1}{2}\pi$.	**M1**
$v = 20\tan\left(\frac{1}{4}\pi - \frac{1}{2}t\right)$.	**A1**
	Total: 9

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Q35

9231/32 • October/November 2025 • Question 7

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Mark scheme.

Must have m: $-mgkv - \mu mg = m\dfrac{\mathrm{d}v}{\mathrm{d}t}$.	**B1**
Separate variables and integrate to a natural logarithm term (ignore modulus signs): $t = -\dfrac{1}{g}\displaystyle\int \dfrac{1}{kv+\mu}\,\mathrm{d}v$, $t = -\dfrac{1}{gk}\ln(kv+\mu)\,[+C]$.	***M1**
Use initial condition to find constant or use correct limits: when $v = u, t = 0$: $C = \dfrac{1}{gk}\ln(kU+\mu)$.	**DM1**
AG, shown convincingly: $t = \dfrac{1}{gk}\ln\left(\dfrac{kU+\mu}{kv+\mu}\right)$. **(b)**	**A1**
Three terms in equation of motion, separate variables and express integrand in a form that can be directly integrated: $-\frac{2}{5}v - 2 = v\dfrac{\mathrm{d}v}{\mathrm{d}x}$, $x = -\frac{5}{2}\displaystyle\int \dfrac{v}{v+5}\,\mathrm{d}v = -\frac{5}{2}\displaystyle\int \left(1 - \dfrac{5}{v+5}\right)\mathrm{d}v$.	**M1**
Correct form: $x = -\frac{5}{2}v + \frac{25}{2}\ln(v+5)\,[+C]$.	**M1**
Use initial condition to find constant or use correct limits: when $x = 0, v = 10$: $C = 25 - \frac{25}{2}\ln(15)$.	**M1**
When $v = 0$: $x = 25 - \frac{25}{2}\ln(15) + \frac{25}{2}\ln(5) = 11.3$ [m]. **(c)**	**A1**
Divide their x from part (b) by the value of t when $v = 0$ using the given answer from part (a): Average speed $= \dfrac{\text{"11.3"}}{\frac{1}{10\times 0.04}\ln\left(\frac{0.04\times 10+0.2}{0.2}\right)} = \dfrac{\text{"11.3"}}{2.5\ln 3}$.	**M1**
$4.10\,[\mathrm{m\,s^{-1}}]$ (accept 4.10–4.11).	**A1**
	Total: 10

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Written by Ali Hashir.

Questions available in the companion questions booklet.