Further Math with Ali Hashir

Equilibrium of a Rigid Body

Topical Questions (CIE 9231)

Compiled from CIE 9231 Further Mathematics past papers (2015–2025), sorted by topic. Each question is numbered for this booklet, with its original paper code and session shown beneath.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

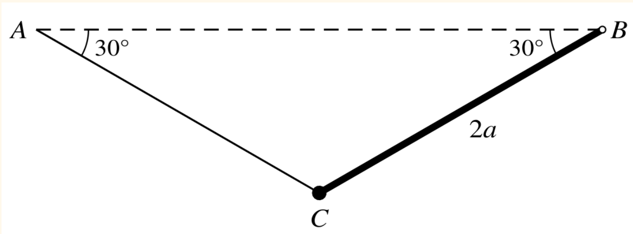
Q1

9231/21, 9231/22 • May/June 2015 • Question 4

A uniform rod BC of length $2a$ and weight W is hinged to a fixed point at B. A particle of weight $3W$ is attached to the rod at C. The system is held in equilibrium by a light elastic string of natural length $\frac{3}{5}a$ in the same vertical plane as the rod. One end of the elastic string is attached to the rod at C and the other end is attached to a fixed point A which is at the same horizontal level as B. The rod and the string each make an angle of $30°$ with the horizontal (see diagram). Find

(i) the modulus of elasticity of the string, [5]

(ii) the magnitude and direction of the force acting on the rod at B. [5]



Q2

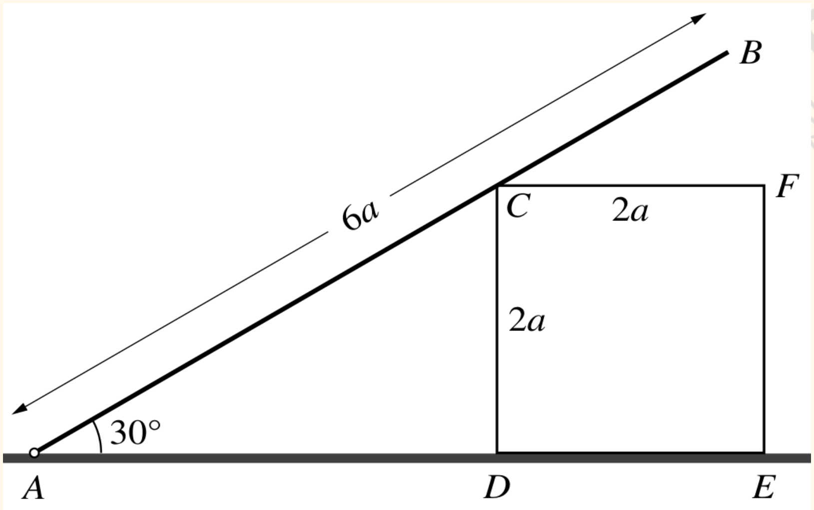
9231/23 • May/June 2015 • Question 4

The diagram shows a central cross-section $CDEF$ of a uniform solid cube of weight W and with edges of length $2a$. The cube rests on a rough horizontal plane. A thin uniform rod AB, of weight W and length $6a$, is hinged to the plane at A. The rod rests in smooth contact with the cube at C, with angle CAD equal to $30°$. The rod is in the same vertical plane as $CDEF$. The coefficient of friction between the plane and the cube is μ. Given that the system is in equilibrium, show that

$$\mu \geqslant \frac{3}{25}\sqrt{3}.$$

[6]

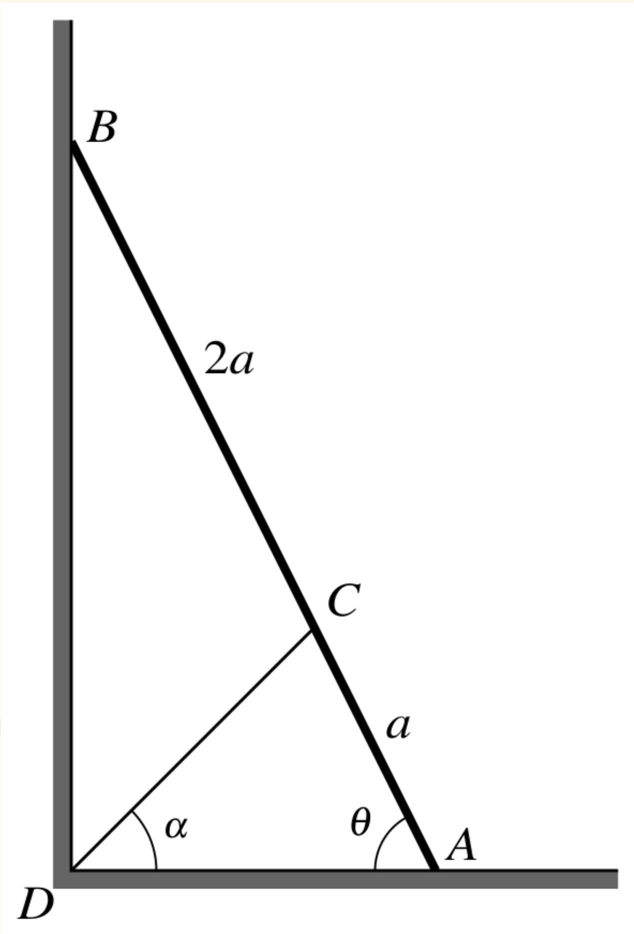
Find the magnitude of the force acting on the rod at A. [4]



Q3

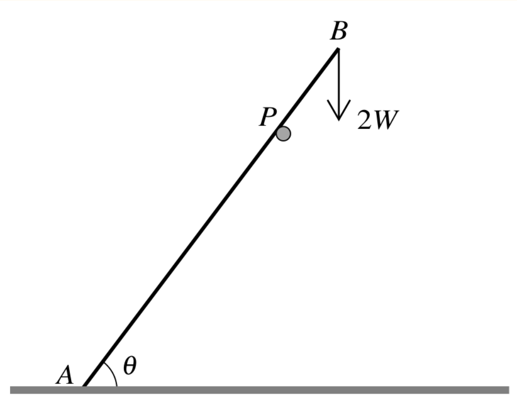
9231/21, 9231/22, 9231/23 • October/November 2015 • Question 1

A uniform ladder AB, of length $3a$ and weight W, rests with the end A in contact with smooth horizontal ground and the end B against a smooth vertical wall. One end of a light inextensible rope is attached to the ladder at the point C, where $AC = a$. The other end of the rope is fixed to the point D at the base of the wall and the rope DC is in the same vertical plane as the ladder AB. The ladder rests in equilibrium in a vertical plane perpendicular to the wall, with the ladder making an angle θ with the horizontal and the rope making an angle α with the horizontal (see diagram). It is given that $\tan\theta = 2\tan\alpha$. Find, in terms of W and α, the tension in the rope and the magnitudes of the forces acting on the ladder at A and at B. [9]



Q4

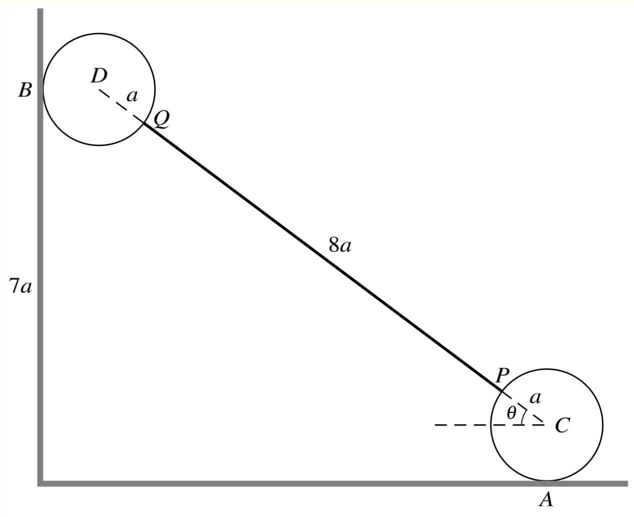
9231/23 • May/June 2016 • Question 2



A uniform rod AB of length $2a$ and weight W rests with its lower end A in contact with a rough horizontal surface. The rod rests on a smooth peg at the point P. A particle of weight $2W$ is suspended from the end B of the rod. The system is in limiting equilibrium with the angle between the rod and the horizontal surface equal to θ, where $\tan\theta = \frac{4}{3}$ (see diagram). The coefficient of friction between the rod and the surface is $\frac{2}{3}$. Find the distance AP. [8]

Q5

9231/21, 9231/22, 9231/23 • October/November 2016 • Question 3



The end P of a uniform rod PQ, of weight kW and length $8a$, is rigidly attached to a point on the surface of a uniform sphere with centre C, weight W and radius a. The end Q is rigidly attached to a point on the surface of an identical sphere with centre D. The points C, P, Q and D are in a straight line. The object consisting of the rod and two spheres rests with one sphere in contact with a rough horizontal surface, at the point A, and the other sphere in contact with a smooth vertical wall, at the point B. The angle between CD and the horizontal is θ. The point B is at a height of $7a$ above the base of the wall (see diagram). The points A, B, C, D, P and Q are all in the same vertical plane.

(i) Show that $\sin\theta = \frac{3}{5}$. [1]

The object is in limiting equilibrium and the coefficient of friction at A is μ.

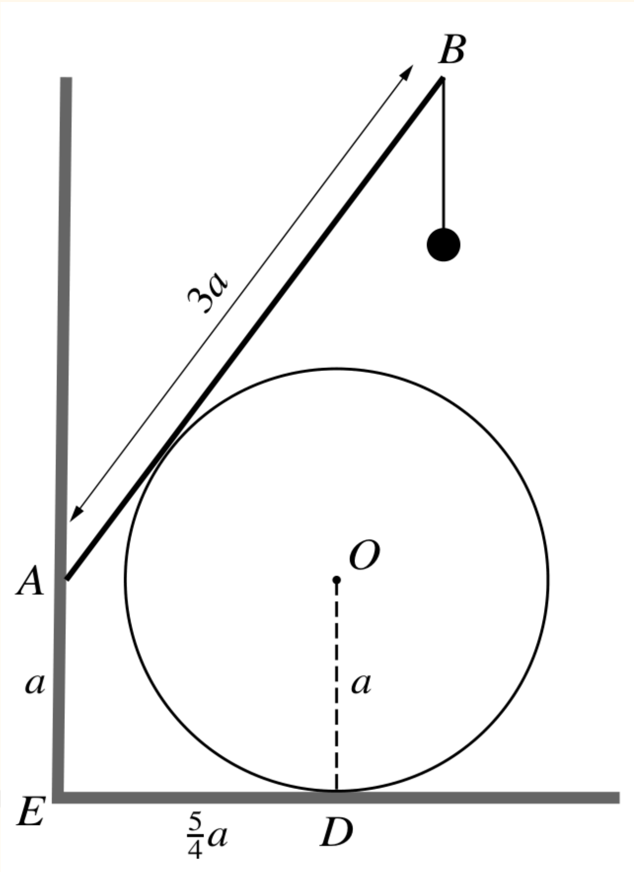
(ii) Find the numerical value of μ. [7]

(iii) Given that the resultant force on the object at A is $W\sqrt{65}$, show that $k = 5$. [3]

Q6

9231/21, 9231/22 • May/June 2017 • Question 2

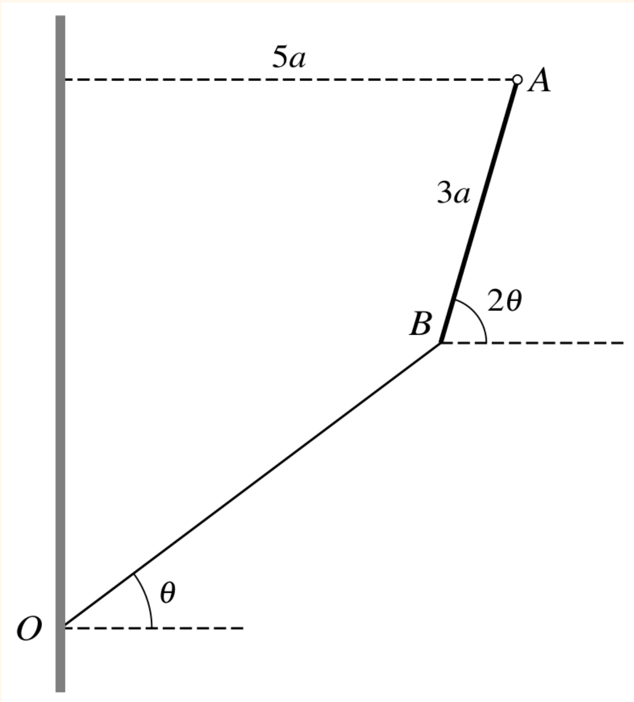
A uniform smooth disc with centre O and radius a is fixed at the point D on a horizontal surface. A uniform rod of length $3a$ and weight W rests on the disc with its end A in contact with a rough vertical wall. The rod and the disc lie in a vertical plane that is perpendicular to the wall. The wall meets the horizontal surface at the point E such that $AE = a$ and $ED = \frac{5}{4}a$. A particle of weight kW is hung from the rod at B (see diagram). The coefficient of friction between the rod and the wall is $\frac{1}{8}$ and the system is in limiting equilibrium. Find the value of k. [8]



Q7

9231/23 • May/June 2017 • Question 4

A uniform rod AB of length $3a$ and weight W is freely hinged to a fixed point at the end A. The end B is below the level of A and is attached to one end of a light elastic string of natural length $4a$. The other end of the string is attached to a point O on a vertical wall. The horizontal distance between A and the wall is $5a$. The string and the rod make angles θ and 2θ respectively with the horizontal (see diagram). The system is in equilibrium with the rod and the string in the same vertical plane. It is given that $\sin\theta = \frac{3}{5}$ and you may use the fact that $\cos 2\theta = \frac{7}{25}$.



(i) Find the tension in the string in terms of W. [3]

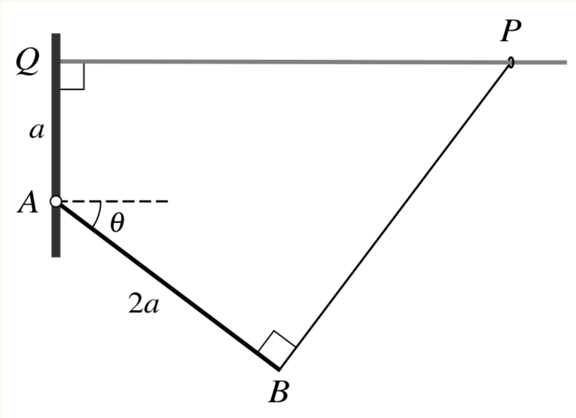
(ii) Find the modulus of elasticity of the string in terms of W. [4]

(iii) Find the angle that the force acting on the rod at A makes with the horizontal. [3]

Q8

9231/21, 9231/22, 9231/23 • October/November 2017 • Question 4

A small ring P of weight W is free to slide on a rough horizontal wire, one end of which is attached to a vertical wall at Q. The end A of a thin uniform rod AB of length $2a$ and weight $\frac{5}{2}W$ is freely hinged to the wall at the point A which is a distance a vertically below Q. A light elastic string of natural length $2a$ has one end attached to the ring P and the other end attached to the rod at B. The string is at right angles to the rod and A, B, P and Q lie in a vertical plane. The system is in limiting equilibrium with AB making an angle θ with the horizontal, where $\sin\theta = \frac{3}{5}$ (see diagram).



(i) Find the tension in the string in terms of W. [2]

(ii) Find the coefficient of friction between the ring and the wire. [2]

(iii) Find the magnitude of the resultant force on the rod at the hinge in terms of W. [3]

(iv) Find the modulus of elasticity of the string in terms of W. [3]

Q9

9231/21, 9231/22 • May/June 2018 • Question 4

A uniform rod AB has length $2a$ and weight W. The end A rests on rough horizontal ground and the end B rests against a smooth vertical wall. The rod is in a vertical plane that is perpendicular to the wall. The angle between the rod and the horizontal is θ. A particle of weight $5W$ hangs from the rod at the point C, with $AC = xa$, where $0 < x < 1$.

(i) By taking moments about A, show that the magnitude of the normal reaction at B is $\dfrac{W(5x+1)}{2\tan\theta}$. [3]

The particle of weight $5W$ is now moved a distance a up the rod, so that $AC = (x+1)a$. This results in the magnitude of the normal reaction at B being double its previous value. The system remains in equilibrium with the rod at angle θ with the horizontal.

(ii) Show that $x = \dfrac{4}{5}$. [3]

The coefficient of friction between the rod and the ground is $\frac{2}{3}$.

(iii) Given that the rod is about to slip when the particle of weight $5W$ is in its second position, find the value of $\tan\theta$. [5]

Q10

9231/23 • May/June 2018 • Question 4

A uniform rod AB has length $2a$ and weight W. The end A rests on rough horizontal ground and the end B rests against a smooth vertical wall. The angle between the rod and the horizontal is θ, where $\tan\theta = \frac{4}{3}$. One end of a light inextensible rope is attached to a point C on the rod. The other end is attached to a point where the vertical wall and the horizontal ground meet. The rope is taut and perpendicular to the rod. The rope and rod are in a vertical plane perpendicular to the wall.

(i) Show that $AC = \dfrac{18}{25}a$. [2]

The magnitude of the frictional force at A is equal to one quarter of the magnitude of the normal reaction force at A.

(ii) Show that the tension in the rope is $\frac{1}{4}W$. [6]

(iii) Find expressions, in terms of W, for the magnitudes of the normal reaction forces at A and B. [2]

Q11

9231/21, 9231/23 • October/November 2018 • Question 4

A uniform rod AB of length $4a$ and weight W is smoothly hinged to a vertical wall at the end A. The rod is held at an angle θ above the horizontal by a light elastic string. One end of the string is attached to the point C on the rod, where $AC = 3a$. The other end of the string is attached to a point D on the wall, with D vertically above A and such that angle $ACD = 2\theta$. A particle of weight $\frac{1}{2}W$ is attached to the rod at B. It is given that $\tan\theta = \frac{8}{15}$.

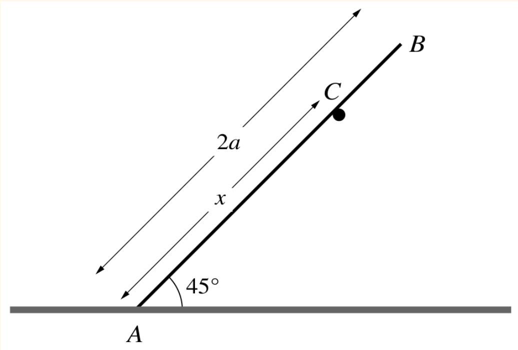
(i) Show that the tension in the string is $\frac{17}{12}W$. [4]

(ii) Find the magnitude and direction of the reaction at the hinge. [5]

(iii) Given that the natural length of the string is $2a$, find its modulus of elasticity. [2]

Q12

9231/22 • October/November 2018 • Question 4



A uniform rod AB of length $2a$ and weight W rests against a smooth horizontal peg at a point C on the rod, where $AC = x$. The lower end A of the rod rests on rough horizontal ground. The rod is in equilibrium inclined at an angle of 45° to the horizontal (see diagram). The coefficient of friction between the rod and the ground is μ. The rod is about to slip at A.

(i) Find an expression for x in terms of a and μ. [5]

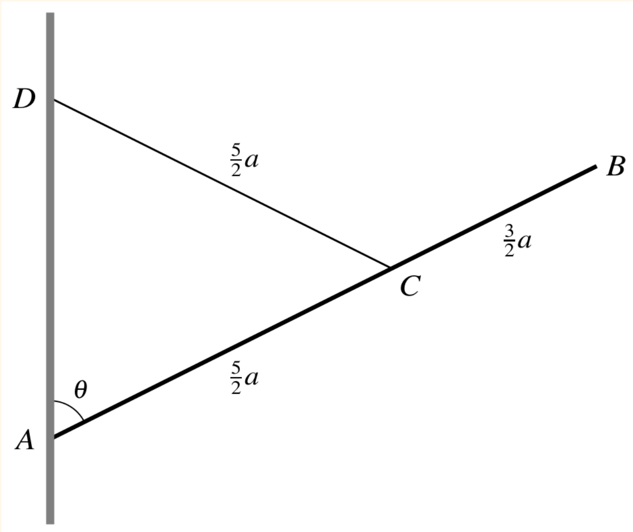
(ii) Hence show that $\mu \geqslant \frac{1}{3}$. [2]

(iii) Given that $x = \frac{3}{2}a$, find the value of μ and the magnitude of the resultant force on the rod at A. [4]

Q13

9231/21, 9231/22 • May/June 2019 • Question 4

A uniform rod AB of length $4a$ and weight W rests with the end A in contact with a rough vertical wall. A light inextensible string of length $\frac{5}{2}a$ has one end attached to the point C on the rod, where $AC = \frac{5}{2}a$. The other end of the string is attached to a point D on the wall, vertically above A. The vertical plane containing the rod AB is perpendicular to the wall. The angle between the rod and the wall is θ, where $\tan\theta = 2$ (see diagram). The end A of the rod is on the point of slipping down the wall and the coefficient of friction between the rod and the wall is μ.

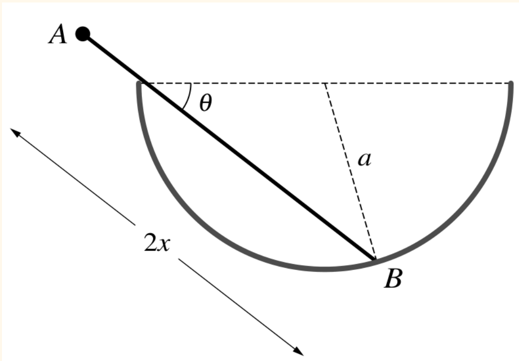


Find, in either order, the tension in the string and the value of μ. [10]

Q14

9231/23 • May/June 2019 • Question 5

A uniform rod AB of length $2x$ and weight W rests on the smooth rim of a fixed hemispherical bowl of radius a. The end B of the rod is in contact with the rough inner surface of the bowl. The coefficient of friction between the rod and the bowl at B is $\frac{1}{3}$. A particle of weight $\frac{1}{4}W$ is attached to the end A of the rod. The end B is about to slip upwards when AB is inclined at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$ (see diagram).



(i) By resolving parallel to the rod, show that the normal component of the reaction of the bowl on the rod at B is $\frac{3}{4}W$. [5]

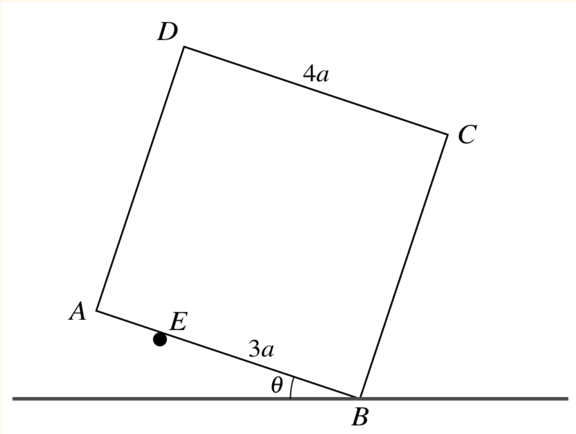
(ii) Find, in terms of W, the reaction between the rod and the smooth rim of the bowl. [4]

(iii) Find x in terms of a. [3]

Q15

9231/21, 9231/22, 9231/23 • October/November 2019 • Question 2

A uniform square lamina $ABCD$ of side $4a$ and weight W rests in a vertical plane with the edge AB inclined at an angle θ to the horizontal, where $\tan\theta = \frac{1}{3}$. The vertex B is in contact with a rough horizontal surface for which the coefficient of friction is μ. The lamina is supported by a smooth peg at the point E on AB, where $BE = 3a$ (see diagram).



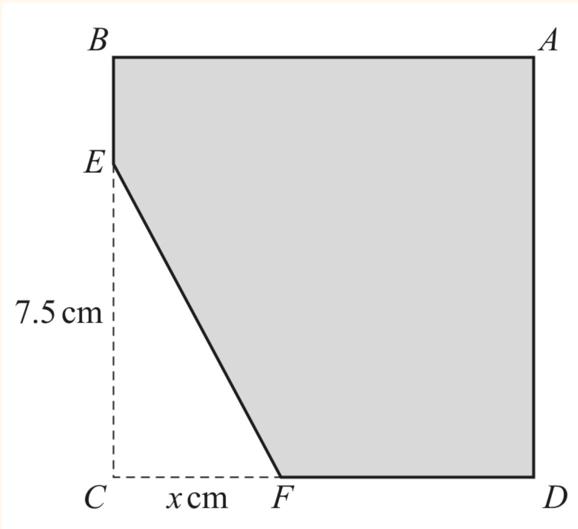
(i) Find expressions in terms of W for the normal reaction forces at E and B. [5]

(ii) Given that the lamina is about to slip, find the value of μ. [3]

Q16

9231/31, 9231/32 • May/June 2020 • Question 4

A uniform square lamina $ABCD$ has sides of length 10cm. The point E is on BC with $EC = 7.5$cm, and the point F is on DC with $CF = x$cm. The triangle EFC is removed from $ABCD$ (see diagram). The centre of mass of the resulting shape $ABEFD$ is a distance $\bar{x}$cm from CB and a distance $\bar{y}$cm from CD.



(a) Show that $\bar{x} = \dfrac{400 - x^2}{80 - 3x}$ and find a corresponding expression for $\bar{y}$. [4]

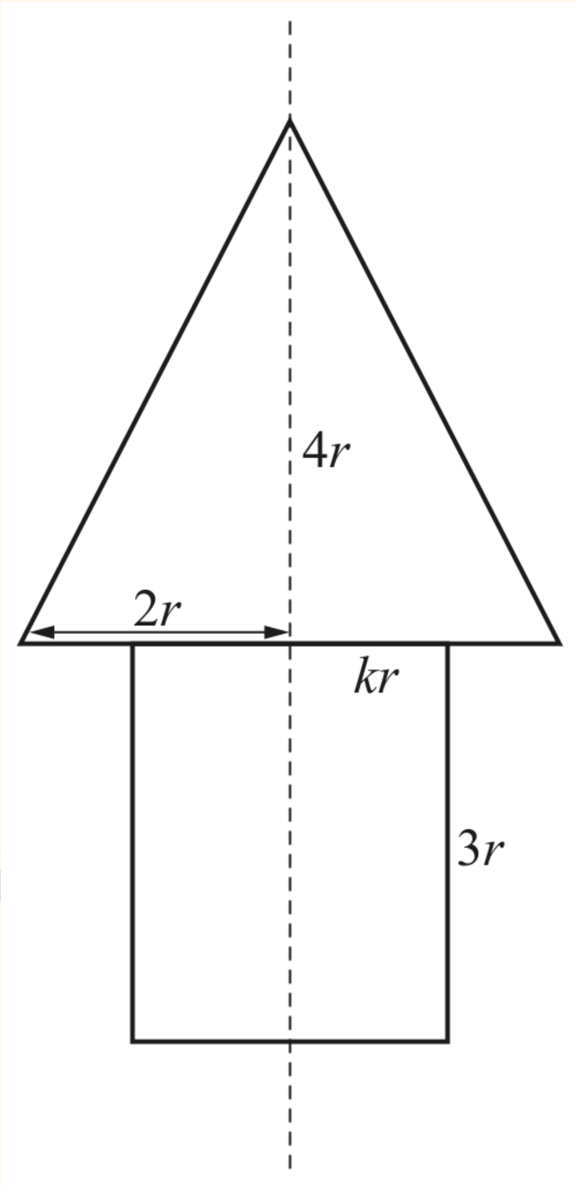
The shape $ABEFD$ is in equilibrium in a vertical plane with the edge DF resting on a smooth horizontal surface.

(b) Find the greatest possible value of x, giving your answer in the form $a + b\sqrt{2}$, where a and b are constants to be determined. [3]

Q17

9231/33 • May/June 2020 • Question 4

A uniform solid circular cone, of vertical height $4r$ and radius $2r$, is attached to a uniform solid cylinder, of height $3r$ and radius kr, where k is a constant less than 2. The base of the cone is joined to one of the circular faces of the cylinder so that the axes of symmetry of the two solids coincide (see diagram). The cone and the cylinder are made of the same material.



(a) Show that the distance of the centre of mass of the combined solid from the vertex of the cone is $\dfrac{(99k^2+96)r}{18k^2+32}$. [4]

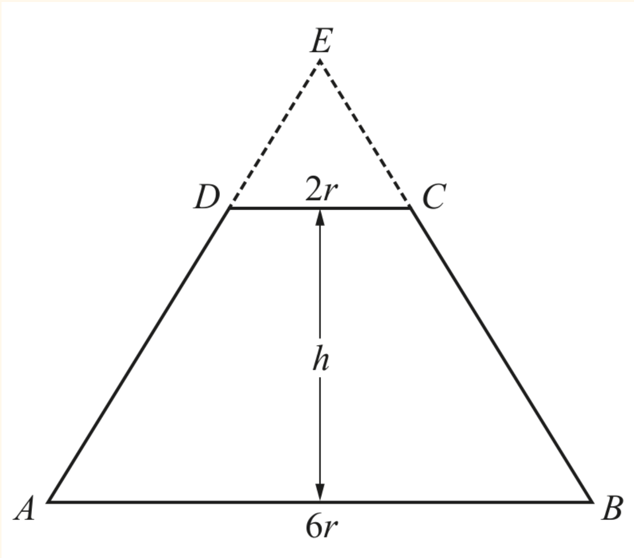
The point C is on the circumference of the base of the cone. When the combined solid is freely suspended from C and hanging in equilibrium, the diameter through C makes an angle α with the downward vertical, where $\tan\alpha = \frac{1}{8}$.

(b) Given that the centre of mass of the combined solid is within the cylinder, find the value of k. [4]

Q18

9231/31, 9231/33 • October/November 2020 • Question 4

The diagram shows the cross-section $ABCD$ of a uniform solid object which is formed by removing a cone with cross-section DCE from the top of a larger cone with cross-section ABE. The perpendicular distance between AB and DC is h, the diameter AB is $6r$ and the diameter DC is $2r$.



(a) Find an expression, in terms of h, for the distance of the centre of mass of the solid object from AB. [4]

The object is freely suspended from the point B and hangs in equilibrium. The angle between AB and the downward vertical through B is θ.

(b) Given that $h = \frac{13}{4}r$, find the value of $\tan\theta$. [2]

Q19

9231/32 • October/November 2020 • Question 3

An object consists of a uniform solid circular cone, of vertical height $4r$ and radius $3r$, and a uniform solid cylinder, of height $4r$ and radius $3r$. The circular base of the cone and one of the circular faces of the cylinder are joined together so that they coincide. The cone and the cylinder are made of the same material.

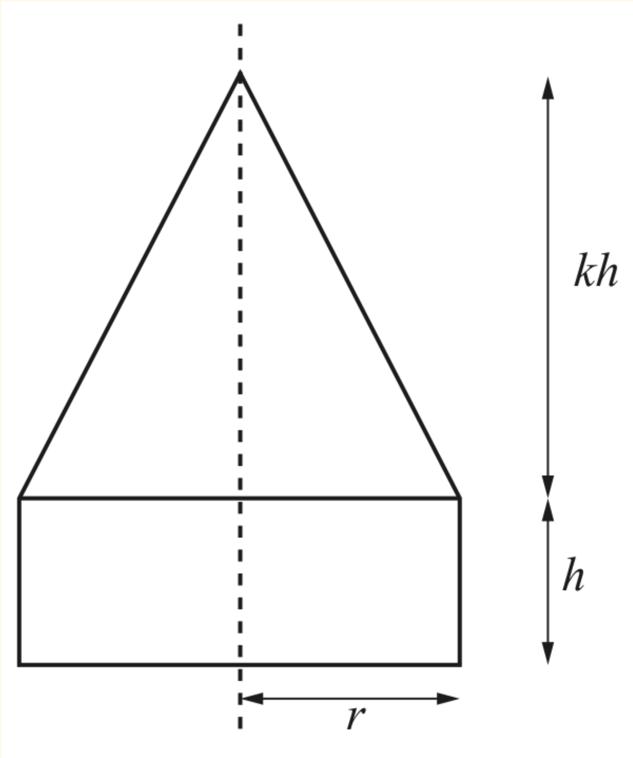
(a) Find the distance of the centre of mass of the object from the end of the cylinder that is not attached to the cone. [4]

(b) Show that the object can rest in equilibrium with the curved surface of the cone in contact with a horizontal surface. [3]

Q20

9231/31, 9231/32 • May/June 2021 • Question 4

A uniform solid circular cone has vertical height kh and radius r. A uniform solid cylinder has height h and radius r. The base of the cone is joined to one of the circular faces of the cylinder so that the axes of symmetry of the two solids coincide (see diagram, which shows a cross-section). The cone and the cylinder are made of the same material.



(a) Show that the distance of the centre of mass of the combined solid from the base of the cylinder is $\dfrac{h(k^2+4k+6)}{4(3+k)}$. [4]

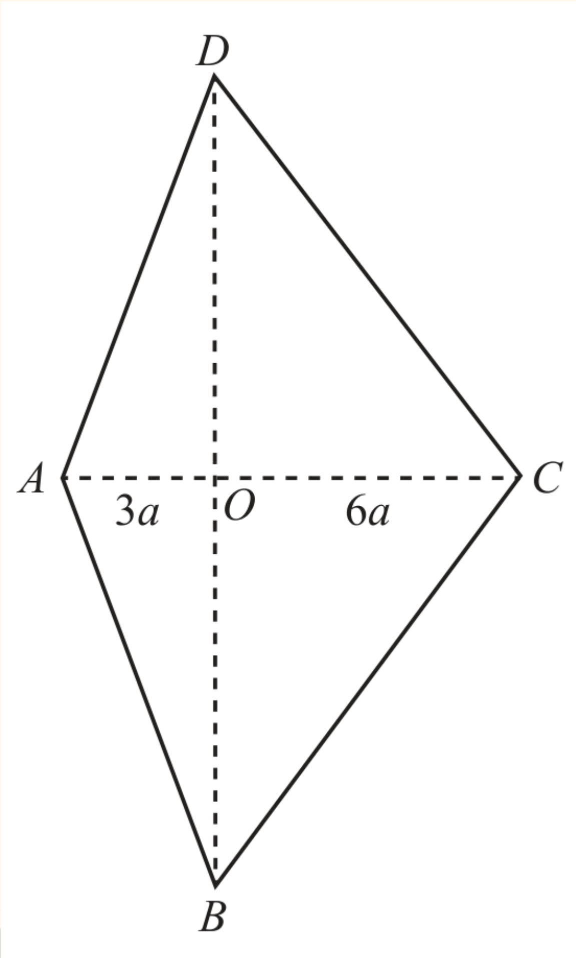
The solid is placed on a plane that is inclined to the horizontal at an angle θ. The base of the cylinder is in contact with the plane. The plane is sufficiently rough to prevent sliding. It is given that $3h = 2r$ and that the solid is on the point of toppling when $\tan\theta = \dfrac{4}{3}$.

(b) Find the value of k. [3]

Q21

9231/33 • May/June 2021 • Question 1

A uniform lamina $ABCD$ consists of two isosceles triangles ABD and BCD. The diagonals of $ABCD$ meet at the point O. The length of AO is $3a$, the length of OC is $6a$ and the length of BD is $16a$ (see diagram).

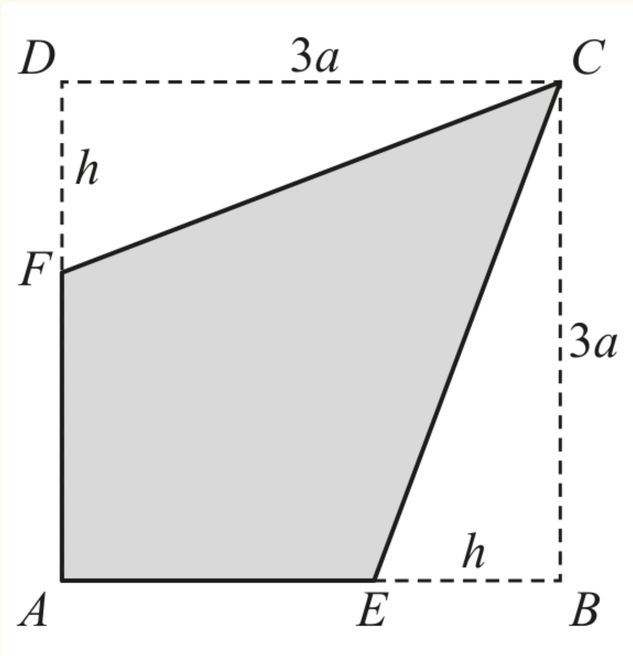


Find the distance of the centre of mass of the lamina from DB. [3]

Q22

9231/31, 9231/33 • October/November 2021 • Question 4

A uniform lamina $AECF$ is formed by removing two identical triangles BCE and CDF from a square lamina $ABCD$. The square has side $3a$ and $EB = DF = h$ (see diagram).



(a) Find the distance of the centre of mass of the lamina $AECF$ from AD and from AB, giving your answers in terms of a and h. [5]

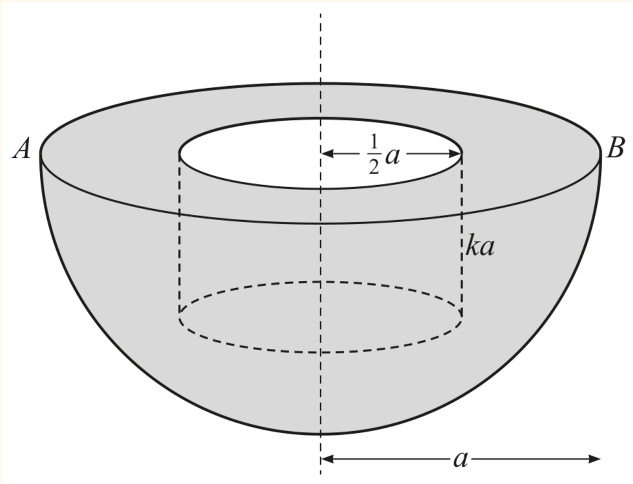
The lamina $AECF$ is placed vertically on its edge AE on a horizontal plane.

(b) Find, in terms of a, the set of values of h for which the lamina remains in equilibrium. [3]

Q23

9231/32 • October/November 2021 • Question 4

An object is formed by removing a solid cylinder, of height ka and radius $\frac{1}{2}a$, from a uniform solid hemisphere of radius a. The axes of symmetry of the hemisphere and the cylinder coincide and one circular face of the cylinder coincides with the plane face of the hemisphere. AB is a diameter of the circular face of the hemisphere (see diagram).



(a) Show that the distance of the centre of mass of the object from AB is $\dfrac{3a(2-k^2)}{2(8-3k)}$. [4]

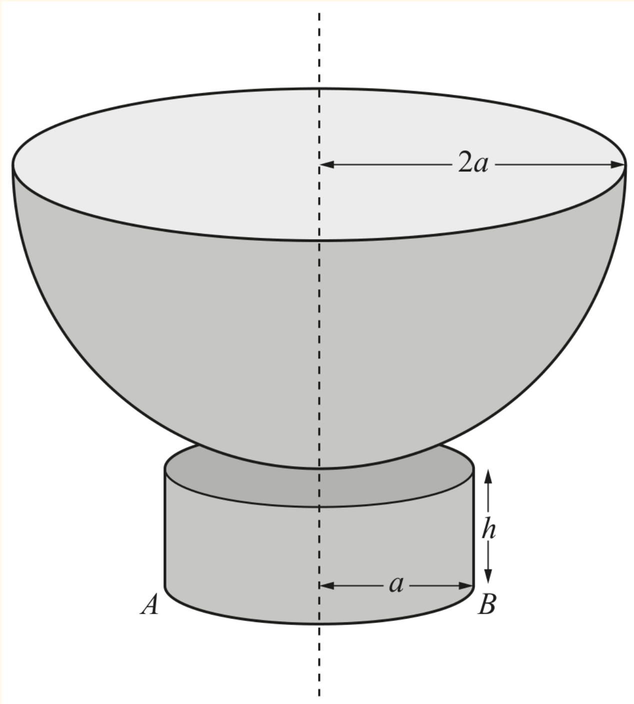
When the object is freely suspended from the point A, the line AB makes an angle θ with the downward vertical, where $\tan\theta = \dfrac{7}{18}$.

(b) Find the possible values of k. [3]

Q24

9231/31, 9231/32 • May/June 2022 • Question 4

An object is composed of a hemispherical shell of radius $2a$ attached to a closed hollow circular cylinder of height h and base radius a. The hemispherical shell and the hollow cylinder are made of the same uniform material. The axes of symmetry of the shell and the cylinder coincide. AB is a diameter of the lower end of the cylinder (see diagram).



(a) Find, in terms of a and h, an expression for the distance of the centre of mass of the object from AB. [4]

The object is placed on a rough plane which is inclined to the horizontal at an angle θ, where $\tan\theta = \dfrac{2}{3}$. The object is in equilibrium with AB in contact with the plane and lying along a line of greatest slope of the plane.

(b) Find the set of possible values of h, in terms of a. [4]

Q25

9231/33 • May/June 2022 • Question 1

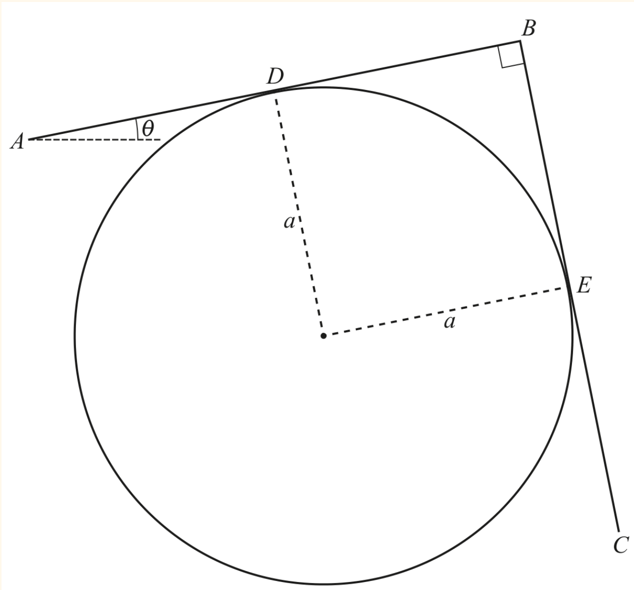
A uniform lamina $OABC$ is a trapezium whose vertices can be represented by coordinates in the x-y plane. The coordinates of the vertices are $O(0, 0)$, $A(15, 0)$, $B(9, 4)$ and $C(3, 4)$.

Find the x-coordinate of the centre of mass of the lamina. [4]

Q26

9231/33 • May/June 2022 • Question 7

A uniform cylinder with a rough surface and of radius a is fixed with its axis horizontal. Two identical uniform rods AB and BC, each of weight W and length $2a$, are rigidly joined at B with AB perpendicular to BC. The rods rest on the cylinder in a vertical plane perpendicular to the axis of the cylinder with AB at an angle θ to the horizontal. D and E are the midpoints of AB and BC respectively and also the points of contact of the rods with the cylinder (see diagram). The rods are about to slip in a clockwise direction. The coefficient of friction between each rod and the cylinder is μ.



The normal reaction between AB and the cylinder is R and the normal reaction between BC and the cylinder is N.

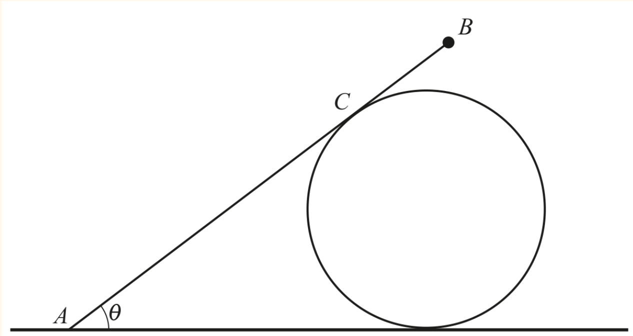
(a) Find the ratio $R : N$ in terms of μ. [6]

(b) Given that $\mu = \dfrac{1}{3}$, find the value of $\tan\theta$. [3]

Q27

9231/31, 9231/33 • October/November 2022 • Question 3

A smooth cylinder is fixed to a rough horizontal surface with its axis of symmetry horizontal. A uniform rod AB, of length $4a$ and weight W, rests against the surface of the cylinder. The end A of the rod is in contact with the horizontal surface. The vertical plane containing the rod AB is perpendicular to the axis of the cylinder. The point of contact between the rod and the cylinder is C, where $AC = 3a$. The angle between the rod and the horizontal surface is θ where $\tan\theta = \frac{3}{4}$ (see diagram). The coefficient of friction between the rod and the horizontal surface is $\frac{6}{7}$.



A particle of weight kW is attached to the rod at B. The rod is about to slip. The normal reaction between the rod and the cylinder is N.

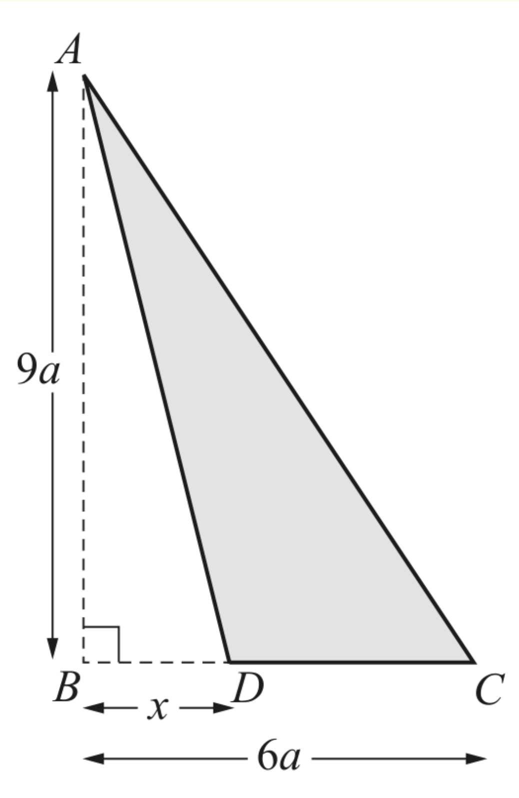
(a) Show that $N = \frac{8}{15}W(1 + 2k)$. [2]

(b) Find the value of k. [5]

Q28

9231/32 • October/November 2022 • Question 2

A uniform lamina is in the form of a triangle ABC in which angle B is a right angle, $AB = 9a$ and $BC = 6a$. The point D is on BC such that $BD = x$ (see diagram). The region ABD is removed from the lamina. The resulting shape ADC is placed with the edge DC on a horizontal surface and the plane ADC is vertical.

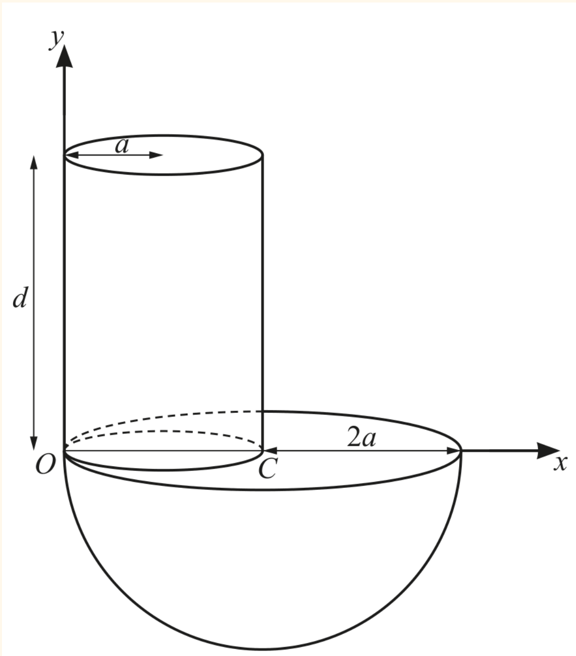


Find the set of values of x, in terms of a, for which the shape is in equilibrium. [6]

Q29

9231/31, 9231/32 • May/June 2023 • Question 4

An object is formed from a solid hemisphere, of radius $2a$, and a solid cylinder, of radius a and height d. The hemisphere and the cylinder are made of the same material. The cylinder is attached to the plane face of the hemisphere. The line OC forms a diameter of the base of the cylinder, where C is the centre of the plane face of the hemisphere and O is common to both circumferences (see diagram). Relative to axes through O, parallel and perpendicular to OC as shown, the centre of mass of the object is $(\bar{x}, \bar{y})$.



(a) Show that $\bar{x} = \dfrac{32a^2 + 3ad}{16a + 3d}$ and find an expression, in terms of a and d, for $\bar{y}$. [5]

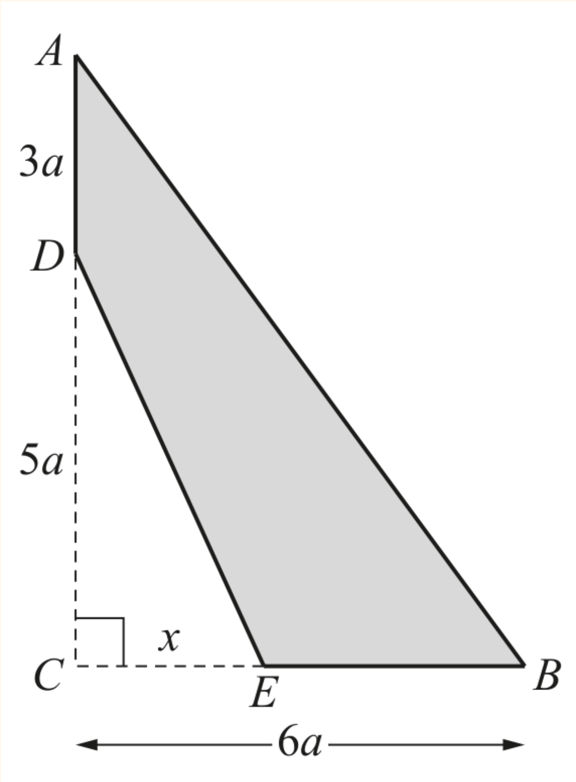
The object is placed on a rough plane which is inclined to the horizontal at an angle θ where $\sin\theta = \dfrac{1}{6}$. The object is in equilibrium with CO horizontal, where CO lies in a vertical plane through a line of greatest slope.

(b) Find d in terms of a. [3]

Q30

9231/33 • May/June 2023 • Question 3

A uniform lamina is in the form of a triangle ABC, with $AC = 8a$, $BC = 6a$ and angle $ACB = 90°$. The point D on AC is such that $AD = 3a$. The point E on CB is such that $CE = x$ (see diagram). The triangle CDE is removed from the lamina.



(a) Find, in terms of a and x, the distance of the centre of mass of the remaining object $ADEB$ from AC. [4]

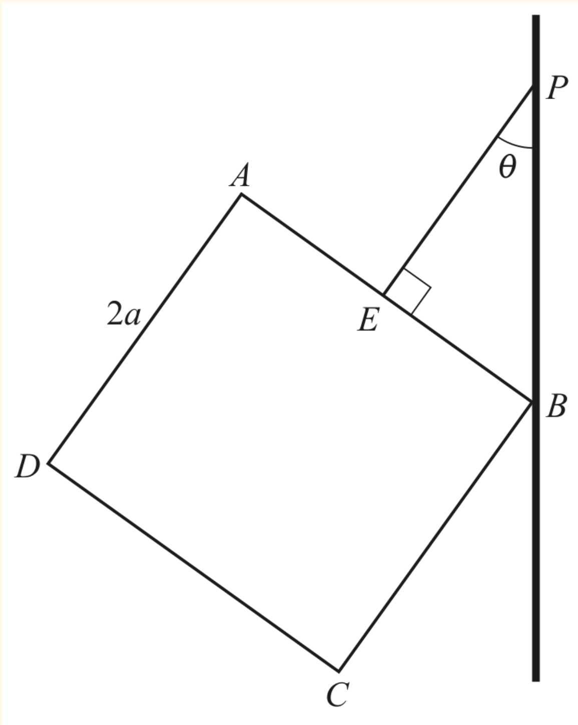
The object $ADEB$ is on the point of toppling about the point E when the object is in the vertical plane with its edge EB on a smooth horizontal surface.

(b) Find x in terms of a. [3]

Q31

9231/31, 9231/33 • October/November 2023 • Question 3

A uniform square lamina of side $2a$ and weight W is suspended from a light inextensible string attached to the midpoint E of the side AB. The other end of the string is attached to a fixed point P on a rough vertical wall. The vertex B of the lamina is in contact with the wall. The string EP is perpendicular to the side AB and makes an angle θ with the wall (see diagram). The string and the lamina are in a vertical plane perpendicular to the wall. The coefficient of friction between the wall and the lamina is $\frac{1}{2}$.

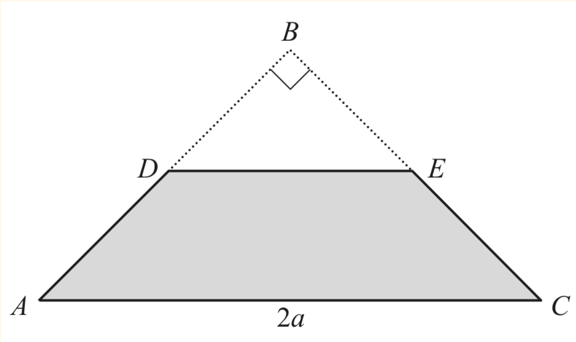


Given that the vertex B is about to slip up the wall, find the value of $\tan\theta$. [8]

Q32

9231/32 • October/November 2023 • Question 3

A uniform lamina is in the form of an isosceles triangle ABC in which $AC = 2a$ and angle $ABC = 90°$. The point D on AB is such that the ratio $DB : AB = 1 : k$. The point E on CB is such that DE is parallel to AC. The triangle DBE is removed from the lamina (see diagram).



(a) Find, in terms of k, the distance of the centre of mass of the remaining lamina $ADEC$ from the midpoint of AC. [4]

When the lamina $ADEC$ is freely suspended from the vertex A, the edge AC makes an angle θ with the downward vertical, where $\tan\theta = \frac{5}{18}$.

(b) Find the value of k. [3]

Q33

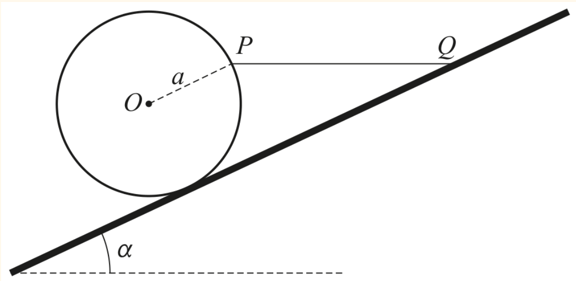
9231/31, 9231/32 • May/June 2024 • Question 4

A ring of weight W, with radius a and centre O, is at rest on a rough surface that is inclined to the horizontal at an angle α where $\tan\alpha = \frac{1}{2}$. The plane of the ring is perpendicular to the inclined surface and parallel to a line of greatest slope of the surface. The point P on the circumference of the ring is such that OP is parallel to the surface.

A light inextensible string is attached to P and to the point Q, which is on the surface, such that PQ is horizontal (see diagram). The points O, P and Q are in the same vertical plane. The system is in limiting equilibrium and the coefficient of friction between the ring and the surface is μ.

(a) Find, in terms of W, the tension in the string PQ. [4]

(b) Find the value of μ. [3]



Q34

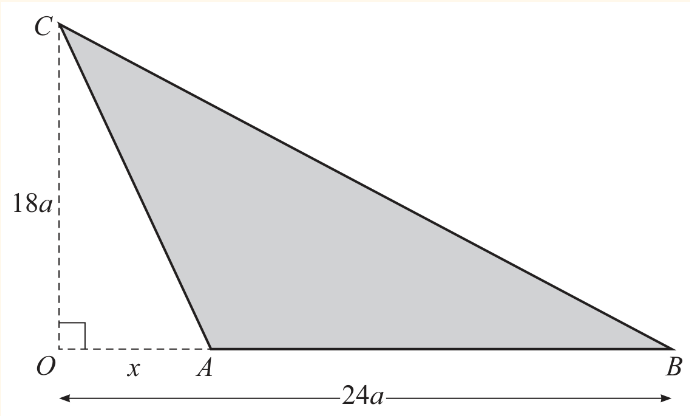
9231/33 • May/June 2024 • Question 5

A uniform lamina is in the form of a triangle OBC, with $OC = 18a$, $OB = 24a$ and angle $COB = 90°$. The point A on OB is such that $OA = x$ (see diagram). The triangle OAC is removed from the lamina.

(a) Find, in terms of a and x, the distance of the centre of mass of the remaining object ABC from OC. [3]

The object ABC is suspended from C. In its equilibrium position, the side AB makes an angle θ with the vertical, where $\tan\theta = \frac{6}{5}$.

(b) Find x in terms of a. [4]

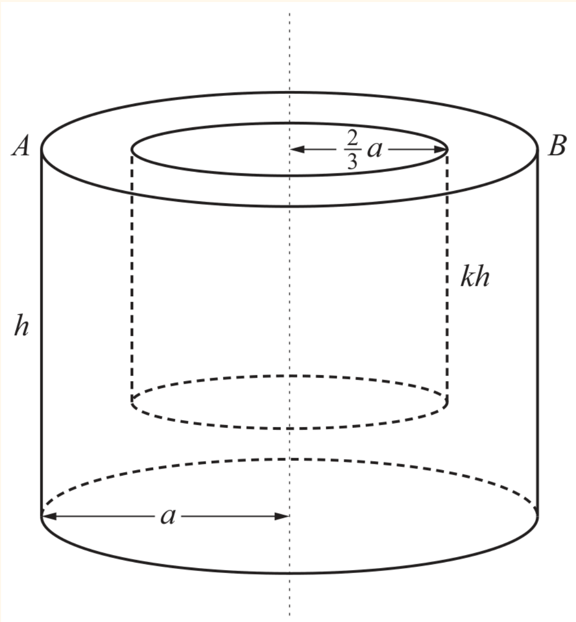


Q35

9231/31, 9231/33 • October/November 2024 • Question 4

An object is formed by removing a cylinder of radius $\frac{2}{3}a$ and height kh $(k < 1)$ from a uniform solid cylinder of radius a and height h. The vertical axes of symmetry of the two cylinders coincide. The upper faces of the two cylinders are in the same plane as each other. The points A and B are the opposite ends of a diameter of the upper face of the object (see diagram).

(a) Find, in terms of h and k, the distance of the centre of mass of the object from AB. [4]



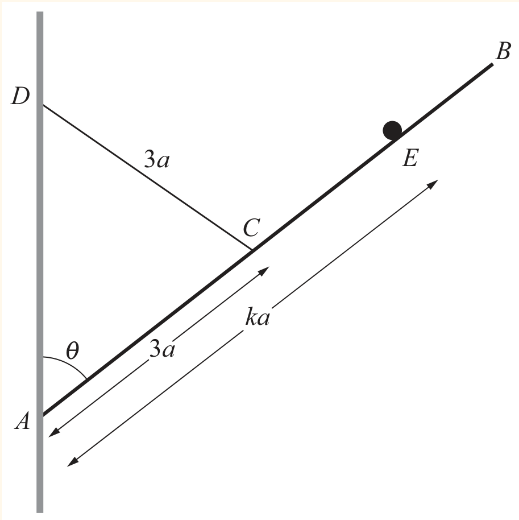
When the object is suspended from A, the angle between AB and the vertical is θ, where $\tan\theta = \dfrac{3}{2}$.

(b) Given that $h = \dfrac{8}{3}a$, find the possible values of k. [3]

Q36

9231/32 • October/November 2024 • Question 4

The end A of a uniform rod AB of length $6a$ and weight W is in contact with a rough vertical wall. One end of a light inextensible string of length $3a$ is attached to the midpoint C of the rod. The other end of the string is attached to a point D on the wall, vertically above A. The rod is in equilibrium when the angle between the rod and the wall is θ, where $\tan\theta = \frac{3}{2}$. A particle of weight W is attached to the point E on the rod, where the distance AE is equal to ka $(3 < k < 6)$ (see diagram). The rod and the string are in a vertical plane perpendicular to the wall. The coefficient of friction between the rod and the wall is $\frac{1}{3}$. The rod is about to slip down the wall.



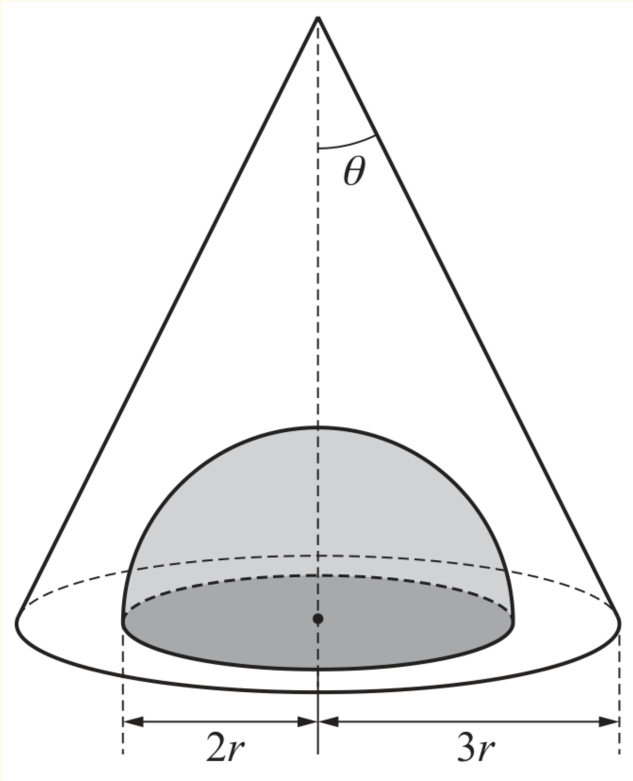
(a) Find the value of k. [5]

(b) Find, in terms of W, the magnitude of the frictional force between the rod and the wall. [2]

Q37

9231/31, 9231/32 • May/June 2025 • Question 4

An object is formed by removing a solid hemisphere, radius $2r$, from a uniform solid cone, radius $3r$ and semi-vertical angle θ, where $\tan\theta = \frac{1}{2}$. The axes of symmetry of the cone and the hemisphere coincide. The base of the cone and the base of the hemisphere are in the same plane as each other (see diagram).



(a) Find, in terms of r, the distance of the centre of mass of the object from its base. [4]

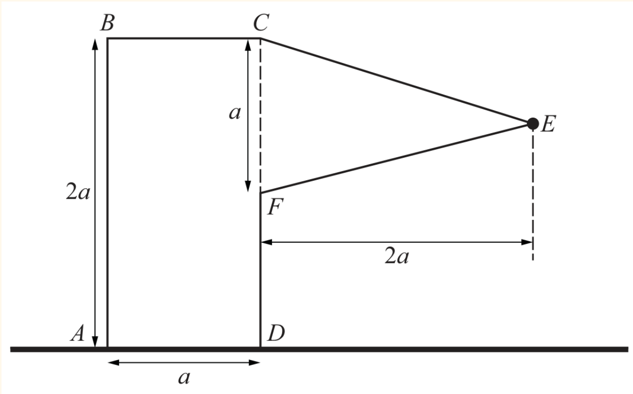
The object is placed such that its circular base makes contact with a rough plane which is inclined to the horizontal at an angle α. The object is on the point of toppling. The plane is sufficiently rough to prevent sliding.

(b) Find the value of α. [3]

Q38

9231/33 • May/June 2025 • Question 4

An object consists of a uniform lamina with a particle attached. The uniform lamina $ABCEFD$ of mass m is formed from a rectangle $ABCD$ and an isosceles triangle CEF, where F is the midpoint of CD. The rectangle has sides $AB = 2a$ and $AD = a$. The triangle CEF has base a and height $2a$. The particle of mass km is attached to the lamina at E. The object rests in a vertical plane with its edge AD on horizontal ground (see diagram).

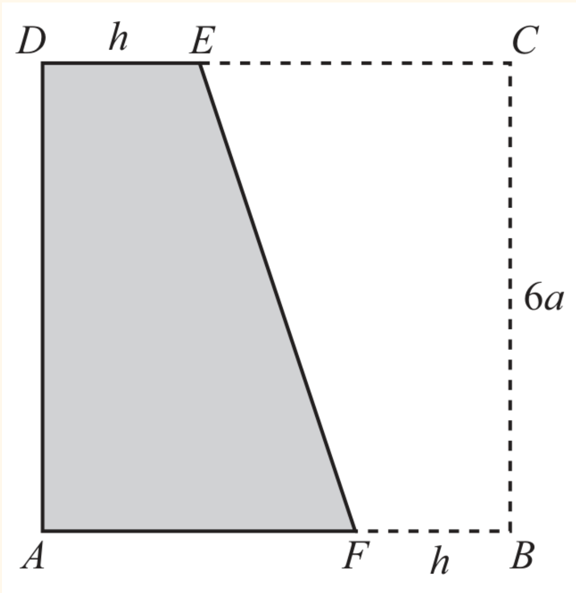


Given that the object is on the point of toppling in its vertical plane about the vertex D, find the value of k. [4]

Q39

9231/34 • May/June 2025 • Question 5

$ABCD$ is a uniform square lamina of side $6a$. Points E and F are on DC and AB respectively and are such that $DE = FB = h$. The quadrilateral $BCEF$ is removed from the square lamina (see diagram).



(a) Show that the distance of the centre of mass of the resulting lamina $AFED$ from AD is $\dfrac{h^2 - 6ah + 36a^2}{18a}$ and find a corresponding expression for the distance of the centre of mass from AB. [5]

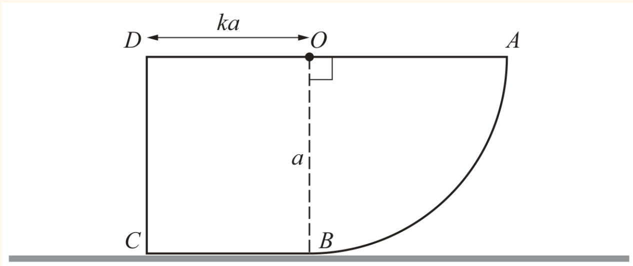
When the lamina $AFED$ is suspended from the point D, the edge DA makes an angle θ with the downward vertical, where $\tan\theta = \frac{7}{15}$.

(b) Find, in terms of a, the two possible values of h. [3]

Q40

9231/32 • October/November 2025 • Question 3

A uniform lamina $OABCD$ is in the form of a rectangle, $OBCD$, joined along the edge OB to a quarter circle OAB. The length of DO is ka and the length of OB is a. The lamina rests in a vertical plane with its edge CB on a horizontal surface (see diagram).



(a) Find, in terms of k, a and π, an expression for the distance of the centre of mass above the horizontal surface. [4]

[You may use without proof the result for the centre of mass of a circular sector in the list of formulae (MF19).]

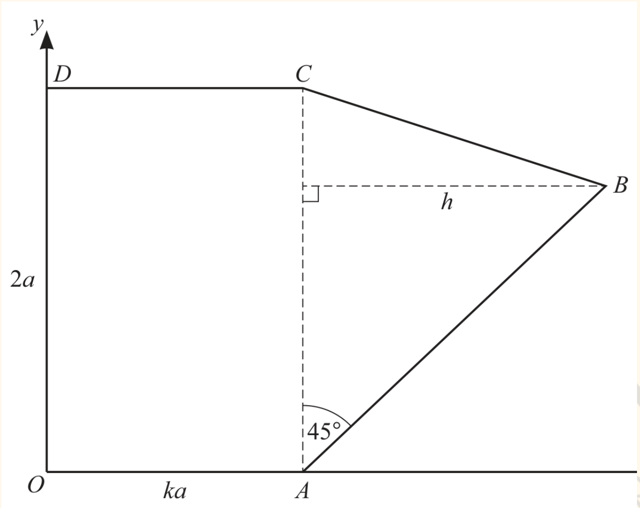
The lamina is on the point of toppling about B.

(b) Find the value of k. [2]

Q41

9231/31, 9231/33 • October/November 2025 • Question 5

A uniform lamina $OABCD$ consists of a rectangle $OACD$ and a triangle ABC. The length of OA is ka, the length of OD is $2a$, the height of triangle ABC is h and angle CAB is 45° (see diagram). Relative to axes through O, parallel and perpendicular to OA as shown, the centre of mass of triangle ABC is $(\bar{x}, \bar{y})$.



(a) Show that $\bar{x}$ is $\frac{1}{3}(3ka + h)$, and find an expression for $\bar{y}$. [3]

The lamina $OABCD$ is placed vertically on its edge OA on a horizontal plane.

(b) Find, in terms of a and k, the set of values of h for which the lamina is in equilibrium. [4]

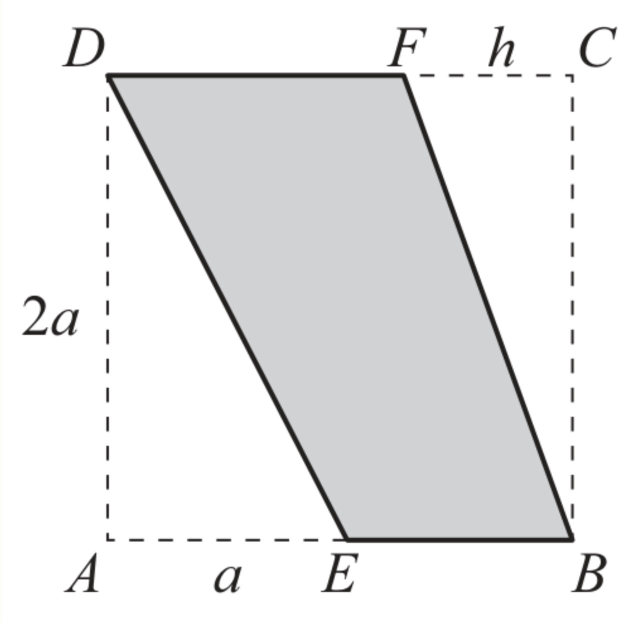
It is now given that $k = \dfrac{\sqrt{3}}{3}$ and that the lamina is on the point of toppling.

(c) Find, in terms of a, the coordinates of the centre of mass of the triangle ABC. [2]

Q42

9231/34 • October/November 2025 • Question 3

The lamina $BFDE$ is obtained by removing triangles AED and BCF from a uniform square lamina $ABCD$ of side $2a$. The length of side AE is a and the length of side FC is h (see diagram). The centre of mass of $BFDE$ is at a distance $\bar{x}$ from AD, and at a distance $\bar{y}$ from AB.



(a) Show that $\bar{x} = \dfrac{h^2 - 6ah + 11a^2}{3(3a - h)}$ and find a corresponding expression for $\bar{y}$. [5]

(b) The lamina $BFDE$ is placed vertically on its edge EB on a smooth horizontal surface.

Find, in terms of a, the set of possible values of h for which the lamina remains in equilibrium. [2]

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Written by Ali Hashir.

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