Further Math with Ali Hashir

Circular Motion

Topical Questions (CIE 9231)

Compiled from CIE 9231 Further Mathematics past papers (2015–2025), sorted by topic. Each question is numbered for this booklet, with its original paper code and session shown beneath.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Q1

9231/21, 9231/22 • May/June 2015 • Question 1

A particle P is moving in a circle of radius 0.25 m. At time t seconds, its velocity is $(2t^2 - 4t + 3)$ m s^{-1}. At a particular time T seconds, where $T > 0$, the magnitude of the transverse component of the acceleration of P is 6 m s^{-2}. Find the magnitude of the radial component of the acceleration of P at this instant. [4]

Q2

9231/21, 9231/22 • May/June 2015 • Question 3

A particle P, of mass m, is placed at the highest point of a fixed solid smooth sphere with centre O and radius a. The particle P is given a horizontal speed u and it moves in part of a vertical circle, with centre O, on the surface of the sphere. When OP makes an angle θ with the upward vertical, and P is still in contact with the surface of the sphere, the speed of P is v and the reaction of the sphere on P has magnitude R. Show that

$$R = mg(3\cos\theta - 2) - \frac{mu^2}{a}.$$

[5]

The particle loses contact with the sphere at the instant when $v = 2u$. Find u in terms of a and g. [4]

Q3

9231/21, 9231/22, 9231/23 • October/November 2015 • Question 4

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. When P is hanging at rest vertically below O, it is projected horizontally. In the subsequent motion P completes a vertical circle. The speed of P when it is at its highest point is u. Show that the least possible value of u is $\sqrt{ag}$. [2]

It is now given that $u = \sqrt{ag}$. When P passes through the lowest point of its path, it collides with, and coalesces with, a stationary particle of mass $\frac{1}{4}m$. Find the speed of the combined particle immediately after the collision. [4]

In the subsequent motion, when OP makes an angle θ with the upward vertical the tension in the string is T. Find an expression for T in terms of m, g and θ. [5]

Find the value of $\cos\theta$ when the string becomes slack. [2]

Q4

9231/21, 9231/22 • May/June 2016 • Question 4

A particle P is at rest at the lowest point on the smooth inner surface of a hollow sphere with centre O and radius a. The particle is projected horizontally with speed u and begins to move in a vertical circle on the inner surface of the sphere. The particle loses contact with the sphere at the point A, where OA makes an angle θ with the upward vertical through O. Given that the speed of P at A is $\sqrt{\frac{3}{5}ag}$, find u in terms of a and g. [5]

Find, in terms of a, the greatest height above the level of O achieved by P in its subsequent motion. (You may assume that P achieves its greatest height before it makes any further contact with the sphere.) [5]

Q5

9231/23 • May/June 2016 • Question 1

A particle P is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle moves in complete vertical circles with centre O. The tension in the string when P is at its lowest point is twice the tension in the string when P is at its highest point. Find, in terms of a and g, the greatest speed of P during the motion. [6]

Q6

9231/23 • May/June 2016 • Question 3

A particle P of mass 0.2 kg is moving in a circle of radius 0.5 m. At time t s its velocity is $(t^2 + ct + d)$ m s^{-1}, where c and d are constants. The magnitude of the resultant force acting on P when $t = 3$ is $\frac{2}{5}\sqrt{17}$ N. The transverse component of the acceleration of P when $t = 3$ is 2 m s^{-2}. Find the values of c and d. [8]

Q7

9231/21, 9231/22, 9231/23 • October/November 2016 • Question 4

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held vertically above O with the string taut and then projected horizontally with speed $\sqrt{(\frac{13}{3}ag)}$. It begins to move in a vertical circle with centre O. When P is at its lowest point, it collides with a stationary particle of mass λm. The two particles coalesce.

(i) Show that the speed of the combined particle immediately after the impact is $\frac{5}{\lambda+1}\sqrt{(\frac{1}{3}ag)}$. [4]

In the subsequent motion, the string becomes slack when the combined particle is at a height of $\frac{1}{3}a$ above the level of O.

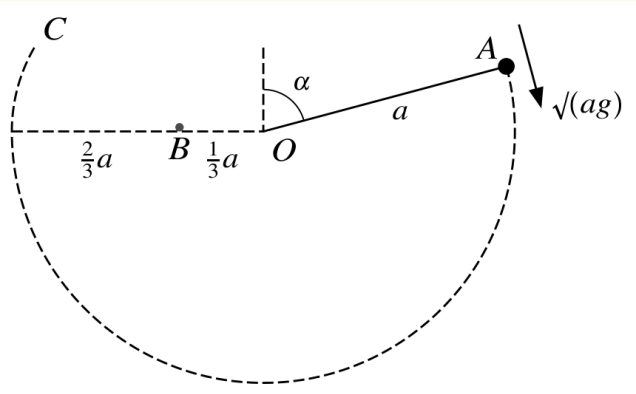
(ii) Find the value of λ. [6]

(iii) Find, in terms of m and g, the instantaneous change in the tension in the string as a result of the collision. [4]

Q8

9231/21, 9231/22 • May/June 2017 • Question 5

A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The point A is such that $OA = a$ and OA makes an angle α with the upward vertical through O. The particle is held at A and then projected downwards with speed $\sqrt{(ag)}$ so that it begins to move in a vertical circle with centre O. There is a small smooth peg at the point B which is at the same horizontal level as O and at a distance $\frac{1}{3}a$ from O on the opposite side of O to A (see diagram).



(i) Show that, when the string first makes contact with the peg, the speed of the particle is $\sqrt{(ag(1 + 2\cos\alpha))}$. [2]

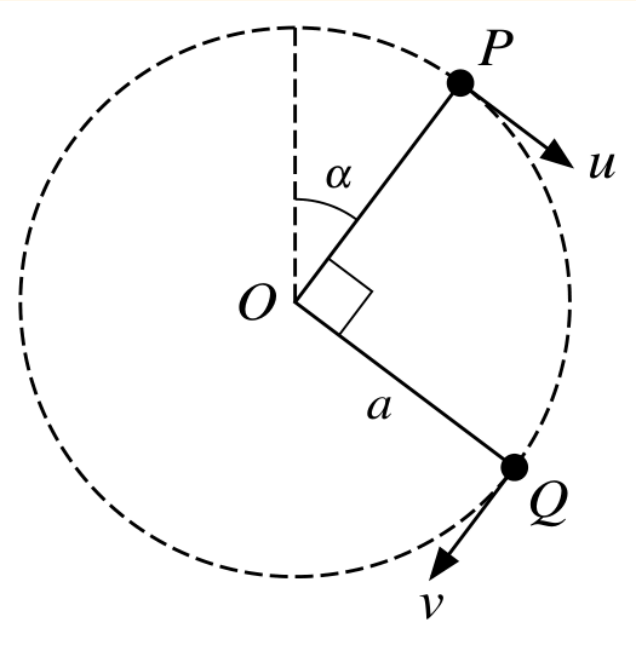
The particle now begins to move in a vertical circle with centre B. When the particle is at the point C where angle $CBO = 150°$, the tension in the string is the same as it was when the particle was at the point A.

(ii) Find the value of $\cos\alpha$. [10]

Q9

9231/23 • May/June 2017 • Question 5

A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is moving in complete vertical circles with the string taut. When the particle is at the point P, where OP makes an angle α with the upward vertical through O, its speed is u. When the particle is at the point Q, where angle $QOP = 90°$, its speed is v (see diagram). It is given that $\cos\alpha = \frac{4}{5}$.



(i) Show that $v^2 = u^2 + \frac{14}{5}ag$. [2]

The tension in the string when the particle is at Q is twice the tension in the string when the particle is at P.

(ii) Obtain another equation relating u^2, v^2, a and g, and hence find u in terms of a and g. [5]

(iii) Find the least tension in the string during the motion. [3]

Q10

9231/21, 9231/22, 9231/23 • October/November 2017 • Question 1

A particle P is moving in a circle of radius 0.8 m. At time t s its velocity is $(8-pt+t^2)$ m s^{-1}, where p is a constant. The magnitude of the transverse component of the acceleration of P when $t = 2$ is zero. Find the magnitude of the radial component of the acceleration of P when $t = 2$. [4]

Q11

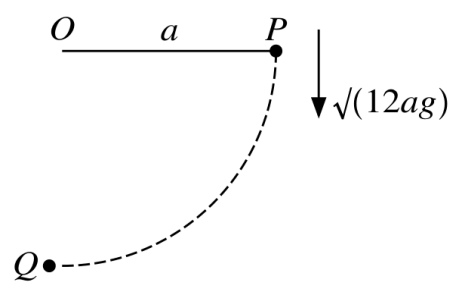
9231/23 • May/June 2018 • Question 1

A particle P is moving in a fixed circle of radius 0.8 m. At time t s its velocity is $(t^2 - t + 2)\,\mathrm{m\,s^{-1}}$. Find the magnitudes of the radial and the transverse components of the acceleration of P when $t = 2$. [3]

Q12

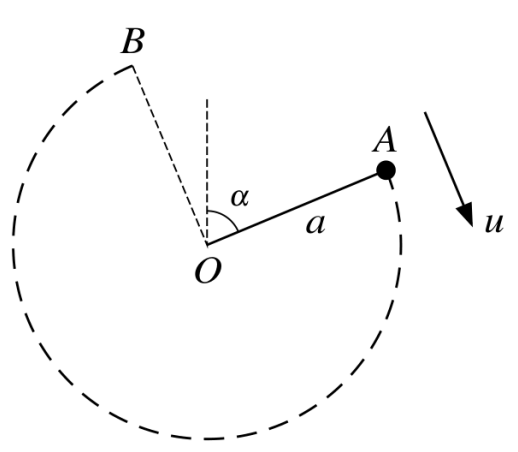
9231/23 • May/June 2018 • Question 5

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is held with the string taut and horizontal. It is projected downwards with speed $\sqrt{12ag}$. At the lowest point of its motion, P collides directly with a particle Q of mass km which is at rest (see diagram). In the collision, P and Q coalesce. The tension in the string immediately after the collision is half of its value immediately before the collision. Find the possible values of k. [11]



Q13

9231/22 • October/November 2018 • Question 3



A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The point A is such that $OA = a$ and OA makes an angle α with the upward vertical, where $\tan\alpha = \frac{12}{5}$. The particle is projected downwards from A with speed u perpendicular to the string and moves in a vertical plane (see diagram). The string becomes slack after the string has rotated through 270° from its initial position, with the particle now at the point B.

(i) Show that $u^2 = 2ag$. [5]

(ii) Find the maximum tension in the string as the particle moves from A to B. [4]

Q14

9231/21, 9231/22 • May/June 2019 • Question 1

A particle P moves along an arc of a circle with centre O and radius 2 m. At time t seconds, the angle POA is θ, where $\theta = 1 - \cos 2t$, and A is a fixed point on the arc of the circle.

(i) Show that the magnitude of the radial component of the acceleration of P when $t = \frac{1}{6}\pi$ is $6\,\mathrm{m\,s^{-2}}$. [2]

(ii) Find the magnitude of the transverse component of the acceleration of P when $t = \frac{1}{6}\pi$. [2]

Q15

9231/23 • May/June 2019 • Question 2

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is moving in a complete vertical circle about O. The points A and B are on the circle, at opposite ends of a diameter, and such that OA makes an acute angle α with the upward vertical through O. The speed of P as it passes through A is $\frac{3}{2}\sqrt{ag}$. The tension in the string when P is at B is four times the tension in the string when P is at A.

(i) Show that $\cos\alpha = \dfrac{3}{4}$. [6]

(ii) Find the tension in the string when P is at B. [2]

Q16

9231/21, 9231/22, 9231/23 • October/November 2019 • Question 1

A particle P is moving in a circle of radius 2 m. At time t seconds, its velocity is $(t-1)^2$ m s^{-1}. At a particular time T seconds, where $T > 0$, the magnitude of the radial component of the acceleration of P is 8 m s^{-2}. Find the magnitude of the transverse component of the acceleration of P at this instant. [5]

Q17

9231/21, 9231/22, 9231/23 • October/November 2019 • Question 4

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O and P is held with the string taut and horizontal. The particle P is projected vertically downwards with speed $\sqrt{2ag}$ so that it begins to move along a circular path. The string becomes slack when OP makes an angle θ with the upward vertical through O.

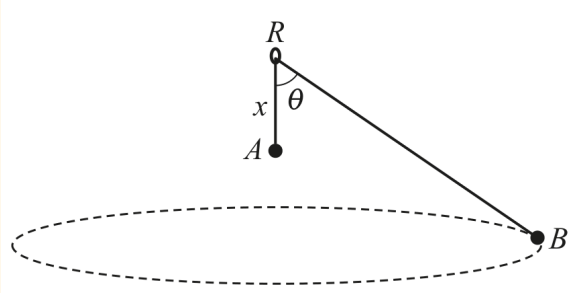
(i) Show that $\cos\theta = \frac{2}{3}$. [5]

(ii) Find the greatest height, above the horizontal through O, reached by P in its subsequent motion. [4]

Q18

9231/31, 9231/32 • May/June 2020 • Question 2

A light inextensible string of length a is threaded through a fixed smooth ring R. One end of the string is attached to a particle A of mass $3m$. The other end of the string is attached to a particle B of mass m. The particle A hangs in equilibrium at a distance x vertically below the ring. The angle between AR and BR is θ (see diagram). The particle B moves in a horizontal circle with constant angular speed $2\sqrt{\frac{g}{a}}$.



Show that $\cos\theta = \frac{1}{3}$ and find x in terms of a. [5]

Q19

9231/31, 9231/32 • May/June 2020 • Question 7

A hollow cylinder of radius a is fixed with its axis horizontal. A particle P, of mass m, moves in part of a vertical circle of radius a and centre O on the smooth inner surface of the cylinder. The speed of P when it is at the lowest point A of its motion is $\sqrt{\frac{7}{2}ga}$.

The particle P loses contact with the surface of the cylinder when OP makes an angle θ with the upward vertical through O.

(a) Show that $\theta = 60°$. [5]

(b) Show that in its subsequent motion P strikes the cylinder at the point A. [5]

Q20

9231/33 • May/June 2020 • Question 1

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O on a smooth horizontal plane. The particle P moves in horizontal circles about O. The tension in the string is $4mg$.

Find, in terms of a and g, the time that P takes to make one complete revolution. [2]

Q21

9231/33 • May/June 2020 • Question 3

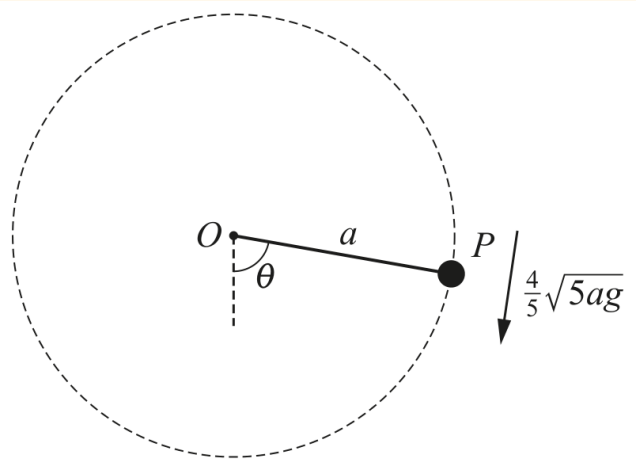
A particle Q of mass m is attached to a fixed point O by a light inextensible string of length a. The particle moves in complete vertical circles about O. The points A and B are on the path of Q with AB a diameter of the circle. OA makes an angle of $60°$ with the downward vertical through O and OB makes an angle of $60°$ with the upward vertical through O. The speed of Q when it is at A is $2\sqrt{ag}$.

Given that T_A and T_B are the tensions in the string at A and B respectively, find the ratio $T_A : T_B$. [6]

Q22

9231/31, 9231/33 • October/November 2020 • Question 2

A particle P is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held with the string taut and making an angle θ with the downward vertical. The particle P is then projected with speed $\frac{4}{5}\sqrt{5ag}$ perpendicular to the string and just completes a vertical circle (see diagram).



Find the value of $\cos\theta$. [5]

Q23

9231/31, 9231/33 • October/November 2020 • Question 3

One end of a light elastic string, of natural length a and modulus of elasticity $4mg$, is attached to a fixed point O. The other end of the string is attached to a particle of mass m. The particle moves in a horizontal circle with a constant angular speed $\sqrt{\frac{g}{a}}$ with the string inclined at an angle θ to the downward vertical through O. The length of the string during this motion is $(k+1)a$.

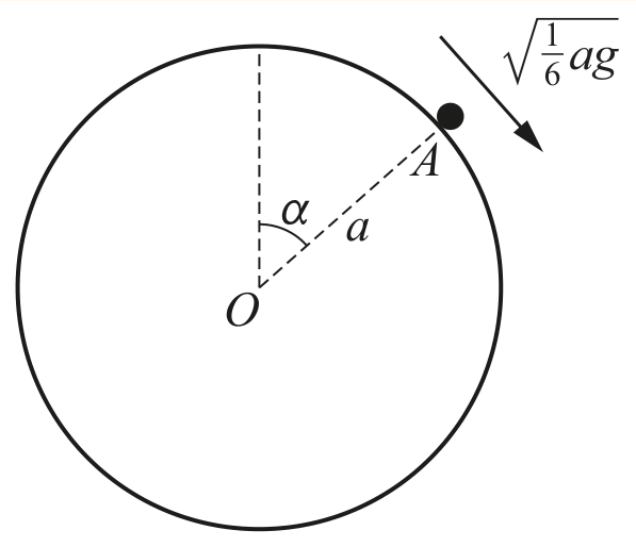
(a) Find the value of k. [4]

(b) Find the value of $\cos\theta$. [2]

Q24

9231/32 • October/November 2020 • Question 1

A fixed smooth solid sphere has centre O and radius a. A particle of mass m is projected downwards with speed $\sqrt{\frac{1}{6}ag}$ from the point A on the surface of the sphere, where OA makes an angle α with the upward vertical through O (see diagram). The particle moves in part of a vertical circle on the surface of the sphere. It loses contact with the sphere at the point B, where OB makes an angle β with the upward vertical through O.



Given that $\cos\alpha = \frac{2}{3}$, find the value of $\cos\beta$. [5]

Q25

9231/32 • October/November 2020 • Question 4

A particle P of mass m is moving in a horizontal circle with angular speed ω on the smooth inner surface of a hemispherical shell of radius r. The angle between the vertical and the normal reaction of the surface on P is θ.

(a) Show that $\cos\theta = \dfrac{g}{\omega^2 r}$. [3]

The plane of the circular motion is at a height x above the lowest point of the shell. When the angular speed is doubled, the plane of the motion is at a height $4x$ above the lowest point of the shell.

(b) Find x in terms of r. [4]

Q26

9231/31, 9231/32 • May/June 2021 • Question 2

A hollow hemispherical bowl of radius a has a smooth inner surface and is fixed with its axis vertical. A particle P of mass m moves in horizontal circles on the inner surface of the bowl, at a height x above the lowest point of the bowl. The speed of P is $\sqrt{\frac{8}{3}ga}$.

Find x in terms of a. [6]

Q27

9231/31, 9231/32 • May/June 2021 • Question 5

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle completes vertical circles with centre O. The points A and B are on the path of P, both on the same side of the vertical through O. OA makes an angle θ with the downward vertical through O and OB makes an angle θ with the upward vertical through O.

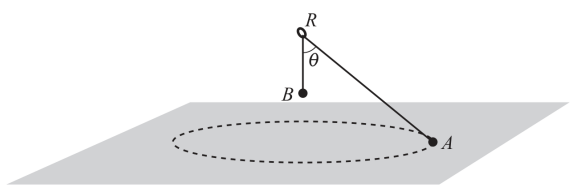
The speed of P when it is at A is u and the speed of P when it is at B is $\sqrt{ag}$. The tensions in the string at A and B are T_A and T_B respectively. It is given that $T_A = 7T_B$.

Find the value of θ and find an expression for u in terms of a and g. [8]

Q28

9231/33 • May/June 2021 • Question 3

Particles A and B, of masses $3m$ and m respectively, are connected by a light inextensible string of length a that passes through a fixed smooth ring R. Particle B hangs in equilibrium vertically below the ring. Particle A moves in horizontal circles on a smooth horizontal surface with speed $\frac{2}{5}\sqrt{ga}$. The angle between AR and BR is θ (see diagram). The normal reaction between A and the surface is $\frac{12}{5}mg$.



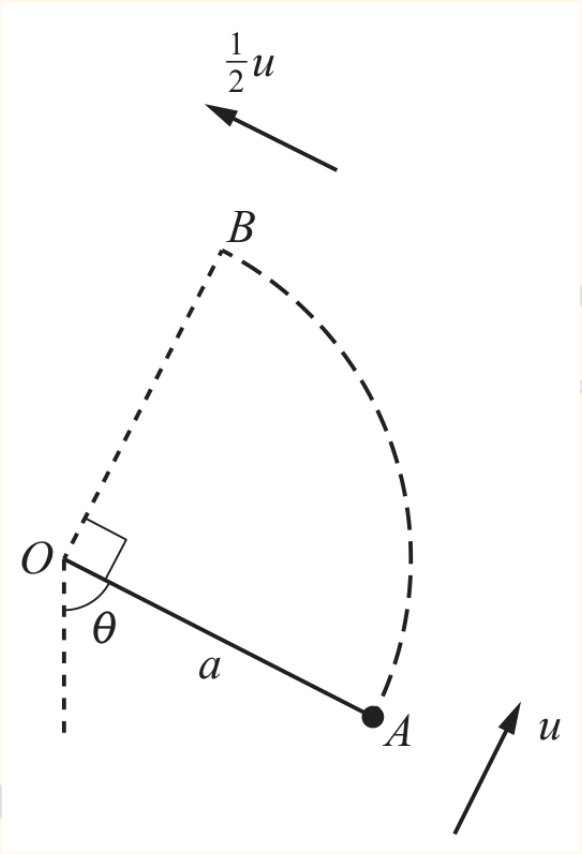
(a) Find $\cos\theta$. [3]

(b) Find, in terms of a, the distance of B below the ring. [3]

Q29

9231/33 • May/June 2021 • Question 4

A particle of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle is initially held with the string taut at the point A, where OA makes an angle θ with the downward vertical through O. The particle is then projected with speed u perpendicular to OA and begins to move upwards in part of a vertical circle. The string goes slack when the particle is at the point B where angle AOB is a right angle. The speed of the particle when it is at B is $\frac{1}{2}u$ (see diagram).



Find the tension in the string at A, giving your answer in terms of m and g. [8]

Q30

9231/31, 9231/33 • October/November 2021 • Question 1

One end of a light elastic string, of natural length a and modulus of elasticity $3mg$, is attached to a fixed point O on a smooth horizontal plane. A particle P of mass m is attached to the other end of the string and moves in a horizontal circle with centre O. The speed of P is $\sqrt{\frac{4}{3}ga}$.

Find the extension of the string. [4]

Q31

9231/31, 9231/33 • October/November 2021 • Question 6

A particle P, of mass m, is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P moves in complete vertical circles about O with the string taut. The points A and B are on the path of P with AB a diameter of the circle. OA makes an angle θ with the downward vertical through O and OB makes an angle θ with the upward vertical through O. The speed of P when it is at A is $\sqrt{5ag}$.

The ratio of the tension in the string when P is at A to the tension in the string when P is at B is $9 : 5$.

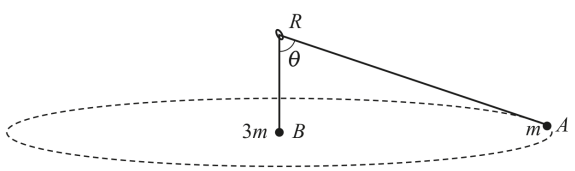
(a) Find the value of $\cos\theta$. [6]

(b) Find, in terms of a and g, the greatest speed of P during its motion. [2]

Q32

9231/32 • October/November 2021 • Question 3

Particles A and B, of masses m and $3m$ respectively, are connected by a light inextensible string of length a that passes through a fixed smooth ring R. Particle B hangs in equilibrium vertically below the ring. Particle A moves in horizontal circles with speed v. Particles A and B are at the same horizontal level. The angle between AR and BR is θ (see diagram).



(a) Show that $\cos\theta = \frac{1}{3}$. [2]

(b) Find an expression for v in terms of a and g. [4]

Q33

9231/32 • October/November 2021 • Question 7

One end of a light inextensible string of length a is attached to a fixed point O. The other end of the string is attached to a particle P of mass m. The particle P is held vertically below O with the string taut and then projected horizontally. When the string makes an angle of $60°$ with the upward vertical, P becomes detached from the string. In its subsequent motion, P passes through the point A which is a distance a vertically above O.

(a) The speed of P when it becomes detached from the string is V. Use the equation of the trajectory of a projectile to find V in terms of a and g. [4]

(b) Find, in terms of m and g, the tension in the string immediately after P is initially projected horizontally. [4]

Q34

9231/31, 9231/32 • May/June 2022 • Question 2

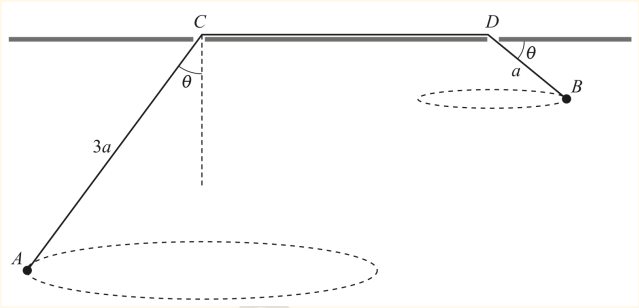
One end of a light inextensible string of length a is attached to a fixed point O. A particle of mass m is attached to the other end of the string. The particle is held at the point A with the string taut. The angle between OA and the downward vertical is equal to α, where $\cos\alpha = \frac{4}{5}$. The particle is projected from A, perpendicular to the string in an upwards direction, with a speed $\sqrt{3ga}$. It then moves along a circular path in a vertical plane. The string first goes slack when it makes an angle θ with the upward vertical through O.

Find the value of $\cos\theta$. [5]

Q35

9231/31, 9231/32 • May/June 2022 • Question 5

A light inextensible string AB passes through two small holes C and D in a smooth horizontal table where $AC = 3a$ and $DB = a$. A particle of mass m is attached at the end A and moves in a horizontal circle with angular velocity ω. A particle of mass $\frac{3}{4}m$ is attached to the end B and moves in a horizontal circle with angular velocity $k\omega$. AC makes an angle θ with the downward vertical and DB makes an angle θ with the horizontal (see diagram).



Find the value of k. [7]

Q36

9231/33 • May/June 2022 • Question 4

One end of a light inextensible string of length a is attached to a fixed point O. A particle of mass m is attached to the other end of the string and is held with the string taut at the point A. At A the string makes an angle θ with the upward vertical through O. The particle is projected perpendicular to the string in a downward direction from A with a speed u. It moves along a circular path in the vertical plane.

When the string makes an angle α with the downward vertical through O, the speed of the particle is $2u$ and the magnitude of the tension in the string is 10 times its magnitude at A.

It is given that $u = \sqrt{\frac{2}{3}ga}$.

(a) Find, in terms of m and g, the magnitude of the tension in the string at A. [6]

(b) Find the value of $\cos\alpha$. [2]

Q37

9231/31, 9231/33 • October/November 2022 • Question 1

A particle of mass 2 kg is attached to one end of a light inextensible string of length 0.6 m. The other end of the string is attached to a fixed point on a smooth horizontal surface. The particle is moving in a circular path on the surface. The tension in the string is 20 N.

Find how many revolutions the particle makes per minute. [3]

Q38

9231/31, 9231/33 • October/November 2022 • Question 5

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The string is held taut with OP horizontal. The particle P is projected vertically downwards with speed $\sqrt{\frac{1}{3}ag}$ and starts to move in a vertical circle. P passes through the lowest point of the circle and reaches the point Q where OQ makes an angle θ with the downward vertical. At Q the speed of P is $\sqrt{kag}$ and the tension in the string is $\frac{11}{6}mg$.

(a) Find the value of k and the value of $\cos\theta$. [4]

At Q the particle P becomes detached from the string.

(b) In the subsequent motion, find the greatest height reached by P above the level of the lowest point of the circle. [4]

Q39

9231/32 • October/November 2022 • Question 1

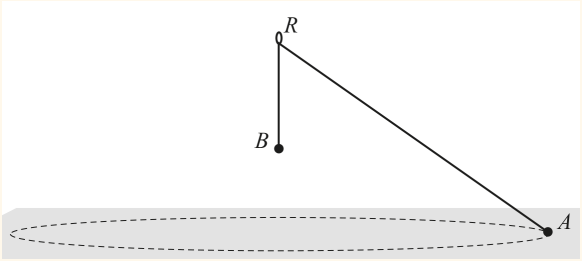
A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The string is held taut with OP making an angle α with the downward vertical, where $\cos\alpha = \frac{2}{3}$. The particle P is projected perpendicular to OP in an upwards direction with speed $\sqrt{3ag}$. It then starts to move along a circular path in a vertical plane.

Find the cosine of the angle between the string and the upward vertical when the string first becomes slack. [4]

Q40

9231/32 • October/November 2022 • Question 6

A light inextensible string is threaded through a fixed smooth ring R which is at a height h above a smooth horizontal surface. One end of the string is attached to a particle A of mass m. The other end of the string is attached to a particle B of mass $\frac{6}{7}m$. The particle A moves in a horizontal circle on the surface. The particle B hangs in equilibrium below the ring and above the surface (see diagram).



When A has constant angular speed ω, the angle between AR and BR is θ and the normal reaction between A and the surface is N.

When A has constant angular speed $\frac{3}{2}\omega$, the angle between AR and BR is α and the normal reaction between A and the surface is $\frac{1}{2}N$.

(a) Show that $\cos\theta = \frac{4}{9}\cos\alpha$. [5]

(b) Find N in terms of m and g and find the value of $\cos\alpha$. [4]

Q41

9231/31, 9231/32 • May/June 2023 • Question 3

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held at the point A, where OA makes an angle θ with the downward vertical through O, and with the string taut. The particle P is projected perpendicular to OA in an upwards direction with speed u. It then starts to move along a circular path in a vertical plane. The string goes slack when P is at B, where angle AOB is $90°$ and the speed of P is $\sqrt{\frac{4}{5}ag}$.

(a) Find the value of $\sin\theta$. [2]

(b) Find, in terms of m and g, the tension in the string when P is at A. [5]

Q42

9231/31, 9231/32 • May/June 2023 • Question 5

A light elastic string of natural length a and modulus of elasticity λmg has one end attached to a fixed point O on a smooth horizontal surface. When a particle of mass m is attached to the free end of the string, it moves with speed v in a horizontal circle with centre O and radius x. When, instead, a particle of mass $2m$ is attached to the free end of the string, this particle moves with speed $\frac{1}{2}v$ in a horizontal circle with centre O and radius $\frac{3}{4}x$.

(a) Find x in terms of a. [5]

(b) Given that $v = \sqrt{12ag}$, find the value of λ. [2]

Q43

9231/33 • May/June 2023 • Question 1

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held at the point A, where OA makes an angle α with the downward vertical through O, and with the string taut. The particle P is projected perpendicular to OA in an upwards direction with speed $\sqrt{3ag}$. It then starts to move along a circular path in a vertical plane. The string goes slack when P is at B, where OB makes an angle θ with the upward vertical.

Given that $\cos\alpha = \frac{4}{5}$, find the value of $\cos\theta$. [4]

Q44

9231/33 • May/June 2023 • Question 5

One end of a light elastic string, of natural length $12a$ and modulus of elasticity kmg, is attached to a fixed point O. The other end of the string is attached to a particle of mass m. The particle moves with constant speed $\frac{3}{2}\sqrt{3ag}$ in a horizontal circle with centre at a distance $12a$ below O. The string is inclined at an angle θ to the downward vertical through O.

(a) Find, in terms of a, the extension of the string. [5]

(b) Find the value of k. [3]

Q45

9231/31, 9231/33 • October/November 2023 • Question 6

A particle P of mass m is attached to one end of a light inextensible rod of length $3a$. An identical particle Q is attached to the other end of the rod. The rod is smoothly pivoted at a point O on the rod, where $OQ = x$. The system, of rod and particles, rotates about O in a vertical plane.

At an instant when the rod is vertical, with P above Q, the particle P is moving horizontally with speed u. When the rod has turned through an angle of $60°$ from the vertical, the speed of P is $2\sqrt{ag}$, and the tensions in the two parts of the rod, OP and OQ, have equal magnitudes.

(a) Show that the speed of Q when the rod has turned through an angle of $60°$ from the vertical is $\dfrac{2x}{3a-x}\sqrt{ag}$. [2]

(b) Find x in terms of a. [5]

(c) Find u in terms of a and g. [4]

Q46

9231/32 • October/November 2023 • Question 1

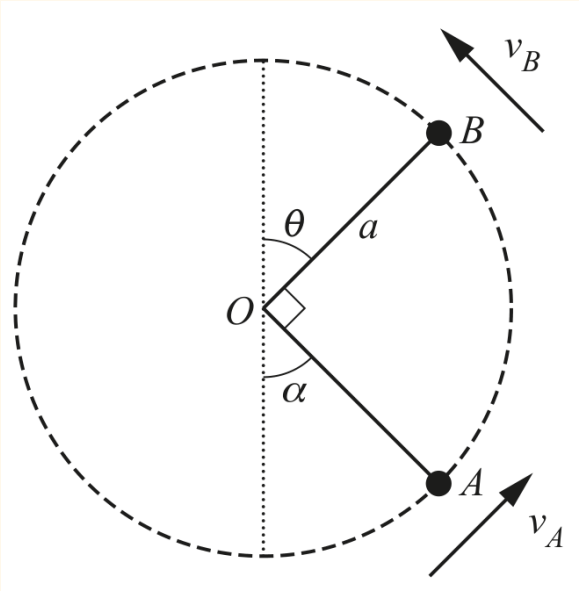
One end of a light inextensible string of length a is attached to a fixed point O. The other end of the string is attached to a particle of mass m. The string is taut and makes an angle θ with the downward vertical through O, where $\cos\theta = \frac{2}{3}$. The particle moves in a horizontal circle with speed v.

Find v in terms of a and g. [4]

Q47

9231/32 • October/November 2023 • Question 5

A bead of mass m moves on a smooth circular wire, with centre O and radius a, in a vertical plane. The bead has speed v_A when it is at the point A where OA makes an angle α with the downward vertical through O, and $\cos\alpha = \frac{3}{5}$. Subsequently the bead has speed v_B at the point B, where OB makes an angle θ with the upward vertical through O. Angle AOB is a right angle (see diagram). The reaction of the wire on the bead at B is in the direction OB and has magnitude equal to $\frac{1}{6}$ of the magnitude of the reaction when the bead is at A.



(a) Find, in terms of m and g, the magnitude of the reaction at B. [6]

(b) Given that $v_A = \sqrt{kag}$, find the value of k. [2]

Q48

9231/31, 9231/32 • May/June 2024 • Question 5

Two particles A and B of masses m and km respectively are connected by a light inextensible string of length a. The particles are placed on a rough horizontal circular turntable with the string taut and lying along a radius of the turntable. Particle A is at a distance a from the centre of the turntable and particle B is at a distance $2a$ from the centre of the turntable. The coefficient of friction between each particle and the turntable is $\frac{1}{5}$.

When the turntable is made to rotate with angular speed $\frac{2}{5}\sqrt{\frac{g}{a}}$, the system is in limiting equilibrium.

(a) Find the tension in the string, in terms of m and g. [4]

(b) Find the value of k. [3]

Q49

9231/33 • May/June 2024 • Question 2

A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity $2mg$. A particle Q of mass km is attached to the other end of the string. Particle P lies on a smooth horizontal table. The string has part of its length in contact with the table and then passes through a small smooth hole H in the table.

Particle P moves in a horizontal circle on the surface of the table with constant speed $\sqrt{\frac{1}{2}ga}$.

Particle Q hangs in equilibrium vertically below the hole with $HQ = \frac{1}{4}a$.

(a) Find, in terms of a, the extension in the string. [4]

(b) Find the value of k. [2]

Q50

9231/33 • May/June 2024 • Question 3

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. When the particle is hanging vertically below O, it is projected horizontally with speed u so that it begins to move along a circular path. When P is at the lowest point of its motion, the tension in the string is T. When OP makes an angle θ with the upward vertical, the tension in the string is S.

(a) Show that $S = T - 3mg(1 + \cos\theta)$. [5]

(b) Given that $u = \sqrt{4ag}$, find the value of $\cos\theta$ when the string goes slack. [2]

Q51

9231/31, 9231/32 • May/June 2024 • Question 7

A smooth sphere with centre O and of radius a is fixed to a horizontal plane. A particle P of mass m is projected horizontally from the highest point of the sphere with speed u, so that it begins to move along the surface of the sphere. The particle P loses contact with the sphere at the point Q on the sphere, where OQ makes an angle θ with the upward vertical through O.

(a) Show that $\cos\theta = \dfrac{u^2 + 2ag}{3ag}$. [4]

It is given that $\cos\theta = \dfrac{5}{6}$.

(b) Find, in terms of a and g, an expression for the vertical component of the velocity of P just before it hits the horizontal plane to which the sphere is fixed. [3]

(c) Find an expression for the time taken by P to fall from Q to the plane. Give your answer in the form $k\sqrt{\dfrac{a}{g}}$, stating the value of k correct to 3 significant figures. [2]

Q52

9231/31, 9231/33 • October/November 2024 • Question 2

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held at the point A with the string taut. It is given that OA makes an angle θ with the downward vertical through O, where $\tan\theta = \frac{3}{4}$. The particle P is projected perpendicular to OA in an upwards direction with speed $\sqrt{5ag}$, and it starts to move along a circular path in a vertical plane. When P is at the point B, where angle AOB is a right angle, the tension in the string is T.

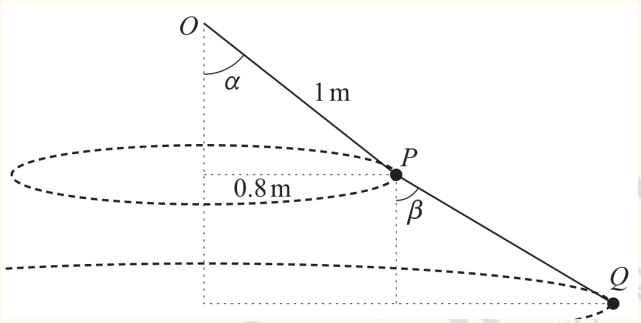
Find T in terms of m and g. [5]

Q53

9231/31, 9231/33 • October/November 2024 • Question 6

A particle P of mass 0.05 kg is attached to one end of a light inextensible string of length 1 m. The other end of the string is attached to a fixed point O. A particle Q of mass 0.04 kg is attached to one end of a second light inextensible string. The other end of this string is attached to P.

The particle P moves in a horizontal circle of radius 0.8 m with angular speed ω rad s^{-1}. The particle Q moves in a horizontal circle of radius 1.4 m also with angular speed ω rad s^{-1}. The centres of the circles are vertically below O, and O, P and Q are always in the same vertical plane. The strings OP and PQ remain at constant angles α and β respectively to the vertical (see diagram).



(a) Find the tension in the string OP. [3]

(b) Find the value of ω. [3]

(c) Find the value of β. [2]

Q54

9231/32 • October/November 2024 • Question 1

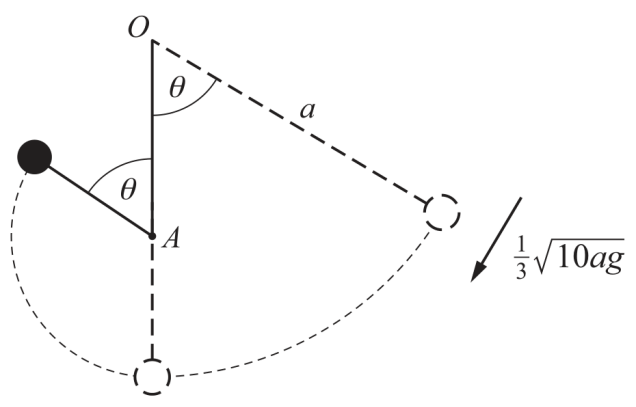
A particle of mass 2 kg is attached to one end of a light elastic string of natural length 0.8 m and modulus of elasticity 100 N. The other end of the string is attached to a fixed point O on a smooth horizontal surface. The particle is moving in a horizontal circle about O with the string taut and with constant angular speed 5 radians per second.

Find the extension of the string. [3]

Q55

9231/32 • October/November 2024 • Question 6

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle P is held with the string taut and the string makes an angle θ with the downward vertical through O. The particle P is projected at right angles to the string with speed $\frac{1}{3}\sqrt{10ag}$ and begins to move downwards along a circular path. When the string is vertical, it strikes a small smooth peg at the point A which is vertically below O. The circular path and the point A are in the same vertical plane. After the string strikes the peg, the particle P begins to move in a vertical circle with centre A. When the string makes an angle θ with the upward vertical through A the string becomes slack (see diagram). The distance of A below O is $\frac{5}{9}a$.



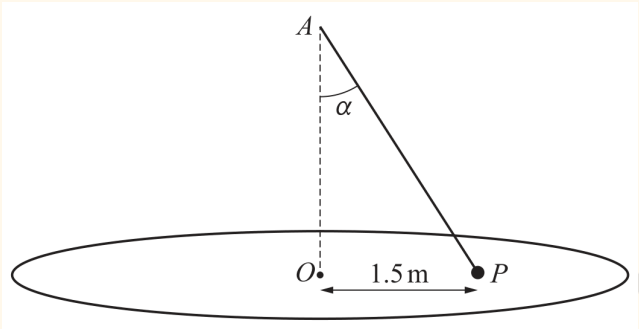
(a) Find the value of $\cos\theta$. [6]

(b) Find the ratio of the tensions in the string immediately before and immediately after it strikes the peg. [4]

Q56

9231/31, 9231/32 • May/June 2025 • Question 3

A rough horizontal disc, centre O, rotates with constant angular speed $\omega\,\text{rad}\,\text{s}^{-1}$. A particle P of mass 1.6 kg lies on the disc at a distance 1.5 m from O, and is attached to a point A vertically above O by a light elastic string. The string has natural length 2 m, modulus of elasticity 32 N and makes an angle α with the vertical OA (see diagram). Particle P moves in a horizontal circle also at a constant angular speed $\omega\,\text{rad}\,\text{s}^{-1}$. Particle P is on the point of slipping in the direction OP. The coefficient of friction between the particle and the disc is 0.5.



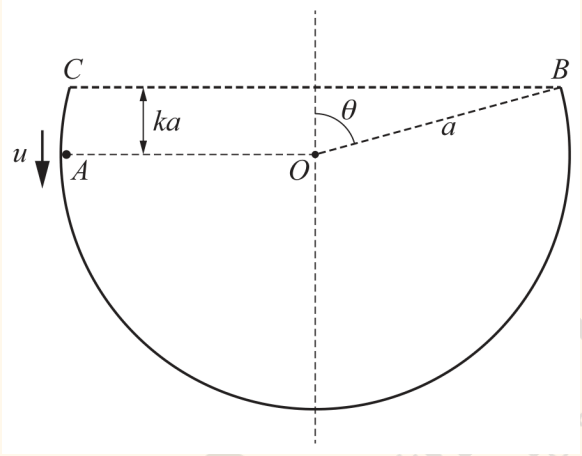
(a) Given that the tension in the string is 8 N, show that $\sin\alpha = 0.6$. [2]

(b) Find the number of revolutions per minute made by the disc and the particle P. [5]

Q57

9231/31, 9231/32 • May/June 2025 • Question 7

A fixed hollow sphere has radius a and centre O. The points A, B and C lie on the inner surface of the sphere with OA and BC horizontal. A portion of the sphere has been removed by a horizontal cut through points B and C at a vertical distance ka above the centre of the sphere, where k is a positive constant and $k < 1$. The points O, A, B and C all lie in the same vertical plane. OB makes an angle θ with the upward vertical through O (see diagram).



A particle P of mass m is free to move on the smooth inner surface of the sphere. The particle P is projected vertically downwards from A with speed u and begins to move in a vertical circle.

(a) In the case where $u = \sqrt{\frac{6}{5}ga}$, the reaction on P at B is half the reaction on P at A. Find the value of k. [5]

(b) Find an expression for u, in terms of a and g, in the case that the particle just reaches B. [1]

(c) Find an expression for u, in terms of a and g, in the case that the particle passes through B and in its subsequent motion reaches C. [4]

Q58

9231/33 • May/June 2025 • Question 1

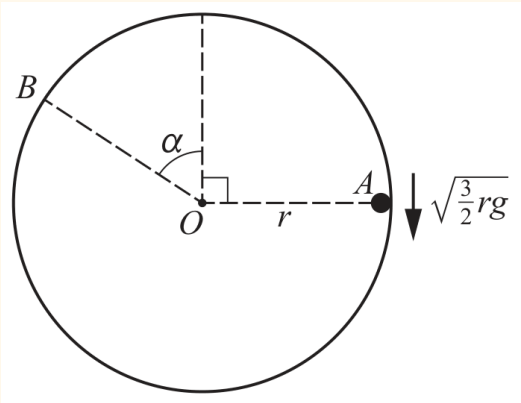
A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle moves in a horizontal circle with constant angular speed ω and with the string inclined at an angle of θ to the downward vertical.

Given that $\tan\theta = \frac{4}{3}$, find ω in terms of a and g. [3]

Q59

9231/33 • May/June 2025 • Question 5

A hollow cylinder of radius r is fixed with its axis horizontal. Points A, B and O are in the same vertical plane perpendicular to the axis of the cylinder, with A and B on the smooth inner surface and O on the axis. OA and OB make angles $90°$ and α respectively with the upward vertical through O, with A and B on opposite sides of the vertical. A particle of mass m is projected vertically downwards from point A with speed $\sqrt{\frac{3}{2}rg}$ and moves in a vertical circle inside the cylinder (see diagram). The particle loses contact with the cylinder at point B.



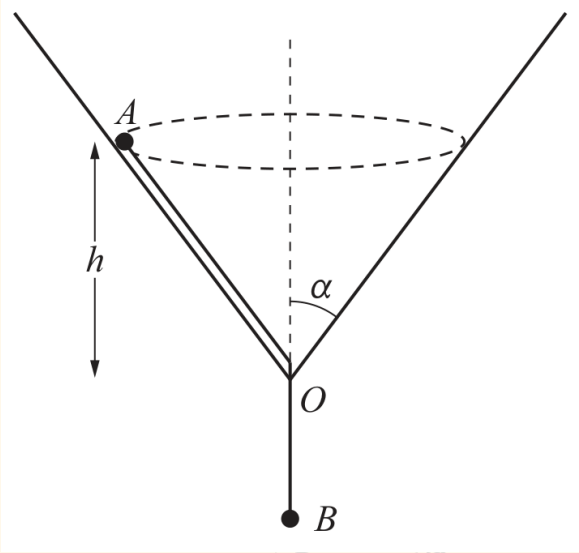
(a) Find the value of α. [4]

(b) In the subsequent motion find, in terms of r, the greatest height above O reached by the particle. [4]

Q60

9231/34 • May/June 2025 • Question 4

A hollow cone with a smooth inner surface is fixed with its vertex O downwards. The semi-vertical angle of the cone is α, where $\tan\alpha = \frac{3}{4}$. A light inextensible string has a particle A of mass m attached to one end and a particle B of mass m attached to the other end. The string passes through a small hole in the cone at O. Particle B hangs in equilibrium below O. Particle A is on the inner surface of the cone at a height h above the level of O and moves in horizontal circles with constant angular speed ω (see diagram).



Find ω in terms of g and h. [6]

Q61

9231/34 • May/June 2025 • Question 7

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle moves in complete vertical circles with centre O, with the string taut. When the string makes an angle θ with the downward vertical through O the speed of P is $\sqrt{4ag}$. The ratio of the greatest and least tensions in the string during the motion is $11 : 1$.

Find the value of $\cos\theta$. [8]

Q62

9231/31, 9231/33 • October/November 2025 • Question 2

A particle P of mass m is moving in a horizontal circle with angular speed ω_1 on the smooth inner surface of a hemispherical shell of radius r. The angle between the upward vertical and the normal reaction of the surface on P is θ_1, where $\tan\theta_1 = \frac{3}{4}$.

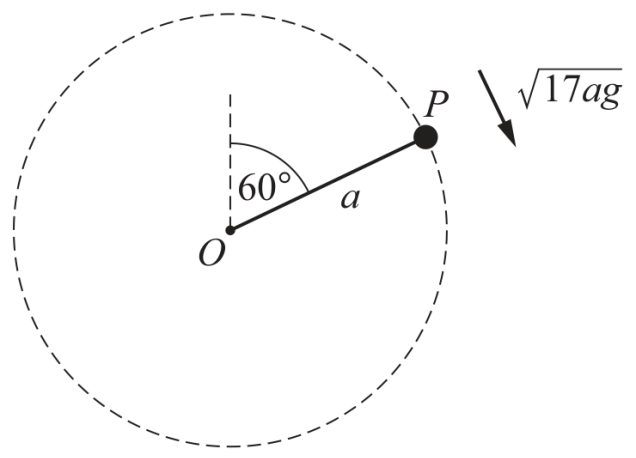
When the angular speed is increased to ω_2, the angle between the upward vertical and the normal reaction of the surface on P becomes θ_2, where $\tan\theta_2 = \frac{4}{3}$.

Find the ratio $\frac{\omega_1}{\omega_2}$. [4]

Q63

9231/31, 9231/33 • October/November 2025 • Question 6

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. Initially P is held with the string taut and making an angle of $60°$ with the upward vertical through O. The particle P is projected perpendicular to the string in a downwards direction with speed $\sqrt{17ag}$. It then starts to move along a circular path in a vertical plane with centre O (see diagram). At the lowest point of its path, vertically below O, the particle P collides with a stationary particle Q.



(a) Find, in terms of a and g, an expression for the speed of P immediately before the collision with Q. [2]

As a result of the collision, P rebounds and moves back along a circular path with centre O. The string becomes slack when P reaches the point on the circle vertically above O.

(b) Find, in terms of a and g, an expression for the speed of P immediately after the collision with Q. [3]

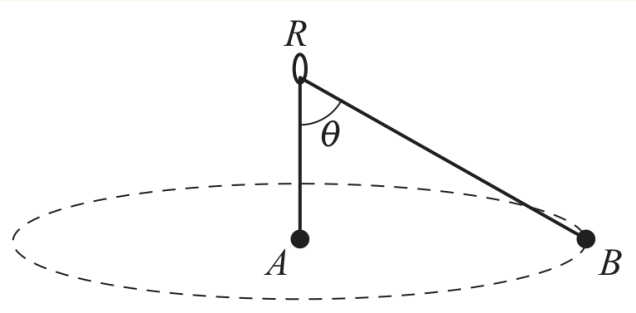
The mass of particle Q is km and the collision between P and Q is perfectly elastic.

(c) Find the value of k. [3]

Q64

9231/34 • October/November 2025 • Question 1

A light inextensible string of length $12a$ is threaded through a fixed smooth ring R. One end of the string is attached to a particle A of mass m. The other end of the string is attached to a particle B of mass $0.5m$. Particle A hangs in equilibrium vertically below the ring. Particle B moves with constant angular speed ω in a horizontal circle with particle A at its centre. The angle between AR and BR is θ (see diagram).

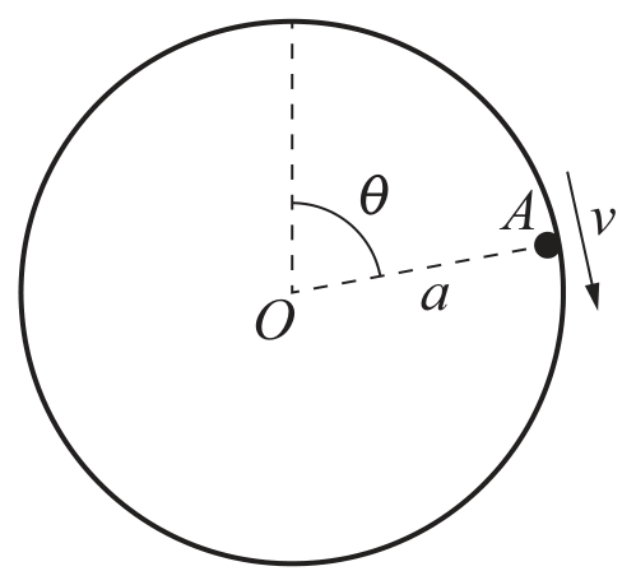


Express ω in terms of g and a. [5]

Q65

9231/34 • October/November 2025 • Question 4

A fixed smooth spherical shell has centre O and radius a. A particle of mass m moves in complete vertical circles on the smooth inner surface of the shell, where the plane of the circular motion is vertical and passes through O. The particle has speed v when it is at point A, where OA makes an angle θ with the upward vertical through O, and $\cos\theta = \frac{1}{18}$ (see diagram).



(a) Show that $v \geq \frac{1}{3}\sqrt{26ag}$. [5]

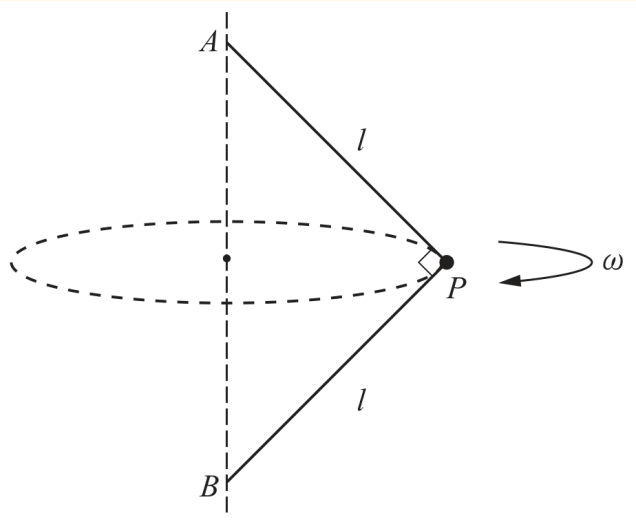
It is given that $v = \frac{1}{3}\sqrt{26ag}$.

(b) Find, in terms of m and g, an expression for the greatest possible value of the normal reaction between the shell and the particle. [3]

Q66

9231/32 • October/November 2025 • Question 1

A particle P of mass m is attached to two light inextensible strings each of length l. The end of one string is attached to a fixed point A and the end of the other string is attached to a fixed point B, with A vertically above B. Angle APB is a right angle. The particle P rotates in a horizontal circle at a constant angular speed ω with both strings taut (see diagram).

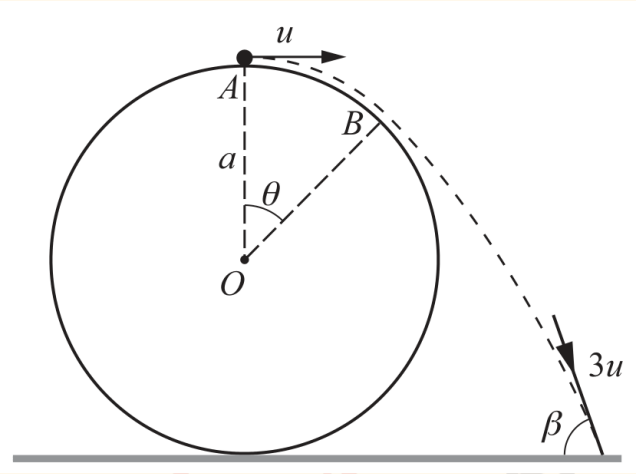


Find the tension in string AP in terms of m, g, l and ω. [4]

Q67

9231/32 • October/November 2025 • Question 6

A fixed smooth sphere with radius a and centre O rests on horizontal ground. A particle is projected horizontally from the highest point, A, of the sphere with speed u. The particle begins to move in a vertical circle along the surface of the sphere. The particle loses contact with the sphere at the point B, where the angle AOB is θ.



After leaving the surface of the sphere, the particle moves freely under gravity before striking the horizontal ground with speed $3u$ at an angle β to the horizontal (see diagram).

Find the value of β. [8]

Answers available in the companion mark scheme booklet.