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Further Math with Ali Hashir

Matrices

Topical Questions (CIE 9231)

Compiled from CIE 9231 Further Mathematics past papers (2015–2025), sorted by topic. Each question is numbered for this booklet, with its original paper code and session shown beneath.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Q1

9231/11, 9231/12 • May/June 2015 • Question 2

Find the value of the constant k for which the system of equations

$$\begin{aligned} 2x - 3y + 4z &= 1, \\ 3x - y &= 2, \\ x + 2y + kz &= 1, \end{aligned}$$

does not have a unique solution. [2]

For this value of k, solve the system of equations. [4]

Q2

9231/11, 9231/12 • May/June 2015 • Question 10

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}.$$

The matrix $\mathbf{A}$ has an eigenvector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Find the corresponding eigenvalue. [2]

The matrix $\mathbf{A}$ also has eigenvalues 4 and 6. Find corresponding eigenvectors. [3]

Hence find a matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix which is to be determined. [2]

The matrix $\mathbf{B}$ is such that $\mathbf{B} = \mathbf{QAQ}^{-1}$, where

$$\mathbf{Q} = \begin{pmatrix} 4 & 11 & 5 \\ 1 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

By using the expression $\mathbf{PDP}^{-1}$ for $\mathbf{A}$, find the set of eigenvalues and a corresponding set of eigenvectors for $\mathbf{B}$. [5]

Q3

9231/13 • May/June 2015 • Question 11

One of the eigenvalues of the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 3 & -4 & 2 \\ -4 & \alpha & 6 \\ 2 & 6 & -2 \end{pmatrix},$$

is -9. Find the value of α. [3]

Find

(i) the other two eigenvalues, λ_1 and λ_2, of $\mathbf{M}$, where $\lambda_1 > \lambda_2$, [5]

(ii) corresponding eigenvectors for all three eigenvalues of $\mathbf{M}$. [3]

It is given that $\mathbf{x} = a\mathbf{e}_1 + b\mathbf{e}_2$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors of $\mathbf{M}$ corresponding to the eigenvalues λ_1 and λ_2 respectively, and a and b are scalar constants. Show that $\mathbf{Mx} = p\mathbf{e}_1 + q\mathbf{e}_2$, expressing p and q in terms of a and b. [3]

Q4

9231/11, 9231/12, 9231/13 • October/November 2015 • Question 6

The matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 10 & -7 & 10 \\ 7 & -5 & 8 \end{pmatrix},$$

has eigenvalues 1 and 3. Find corresponding eigenvectors. [3]

It is given that $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$. Find the corresponding eigenvalue. [2]

Find a diagonal matrix $\mathbf{D}$ and matrices $\mathbf{P}$ and $\mathbf{P}^{-1}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. [5]

Q5

9231/11, 9231/12 • May/June 2016 • Question 10

Write down the eigenvalues of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

and find corresponding eigenvectors. [4]

Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, and hence find the matrix $\mathbf{A}^n$, where n is a positive integer. [8]

Q6

9231/13 • May/June 2016 • Question 3

Find the two values of the constant k for which the equations

$$kx + y + z = 2,$$

$$x + ky + z = -1,$$

$$x + y + kz = -1,$$

have no unique solution. [4]

Show that, for one of these values of k, the equations have no solution, and solve the equations for the other value of k. [3]

Q7

9231/13 • May/June 2016 • Question 11

It is given that 1 and 4 are eigenvalues of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & -3 \\ -8 & 6 & -3 \\ 8 & -2 & 7 \end{pmatrix}.$$

Find eigenvectors corresponding to each of these eigenvalues. [3]

Given further that $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, find the corresponding eigenvalue. [2]

Write down matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix, and find $\mathbf{P}^{-1}$. [5]

Write down a matrix $\mathbf{C}$ such that $\mathbf{C}^2 = \mathbf{D}$, and deduce a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$. [4]

Q8

9231/11, 9231/12, 9231/13 • October/November 2016 • Question 3

Find a matrix $\mathbf{A}$ whose eigenvalues are -1, 1, 2 and for which corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

respectively. [7]

Q9

9231/11, 9231/12 • May/June 2017 • Question 4

(i) Find the value of k for which the set of linear equations

$$x + 3y + kz = 4,$$

$$4x - 2y - 10z = -5,$$

$$x + y + 2z = 1,$$

has no unique solution. [3]

(ii) For this value of k, find the set of possible solutions, giving your answer in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{a} + t\mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are vectors and t is a scalar. [3]

Q10

9231/11, 9231/12 • May/June 2017 • Question 5

The matrix $\mathbf{A}$, given by

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -2 \\ 6 & 4 & -6 \\ 6 & 5 & -7 \end{pmatrix},$$

has eigenvalues 1, -1 and -2.

(i) Find a set of corresponding eigenvectors. [4]

(ii) The matrix $\mathbf{B}$ is given by $\mathbf{B} = \mathbf{A} - 2\mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix. Write down the eigenvalues of $\mathbf{B}$, and state a set of corresponding eigenvectors. [2]

Q11

9231/13 • May/June 2017 • Question 10

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 6 & -8 & 7 \\ 7 & -9 & 7 \\ 6 & -6 & 5 \end{pmatrix}.$$

(i) Given that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, find the corresponding eigenvalue. [2]

(ii) Given also that -1 is an eigenvalue of $\mathbf{A}$, find a corresponding eigenvector. [2]

(iii) It is given that the determinant of $\mathbf{A}$ is equal to the product of the eigenvalues of $\mathbf{A}$. Use this result to find the third eigenvalue of $\mathbf{A}$, and find also a corresponding eigenvector. [3]

(iv) Write down matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix, and hence find the matrix $\mathbf{A}^n$ in terms of n, where n is a positive integer. [6]

Q12

9231/11, 9231/12, 9231/13 • October/November 2017 • Question 11

(i) The vector $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{A}$, with corresponding eigenvalue λ, and is also an eigenvector of the matrix $\mathbf{B}$, with corresponding eigenvalue μ. Show that $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{AB}$ with corresponding eigenvalue $\lambda\mu$. [3]

(ii) Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 3 \\ 3 & 2 & -3 \\ 1 & 1 & 2 \end{pmatrix}.$$

[6]

(iii) The matrix $\mathbf{B}$, where

$$\mathbf{B} = \begin{pmatrix} 3 & 6 & 1 \\ 1 & -2 & -1 \\ 6 & 6 & -2 \end{pmatrix},$$

has eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find the eigenvalues of the matrix $\mathbf{AB}$, and state corresponding eigenvectors. [4]

Q13

9231/11, 9231/12 • May/June 2018 • Question 11

It is given that $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{A}$, with corresponding eigenvalue λ.

(i) Write down another eigenvector of $\mathbf{A}$ corresponding to λ. [1]

(ii) Write down an eigenvector and corresponding eigenvalue of $\mathbf{A}^n$, where n is a positive integer. [2]

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 7 & 0 \\ 4 & 8 & 1 \end{pmatrix}$.

(iii) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^n = \mathbf{PDP}^{-1}$. [7]

(iv) Determine the set of values of the real constant k such that

$$\sum_{n=1}^{\infty} k^n \left(\mathbf{A}^n - k\mathbf{A}^{n+1}\right) = k\mathbf{A}.$$

[4]

Q14

9231/13 • May/June 2018 • Question 5

It is given that $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{A}$ with corresponding eigenvalue λ.

(i) Show that $\mathbf{e}$ is an eigenvector of $\mathbf{A}^3$ and state the corresponding eigenvalue. [3]

It is given that

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}.$$

(ii) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{A}^3 + \mathbf{I} = \mathbf{PDP}^{-1},$$

where $\mathbf{I}$ is the 2×2 identity matrix. [5]

Q15

9231/11, 9231/13 • October/November 2018 • Question 5

It is given that λ is an eigenvalue of the matrix $\mathbf{A}$ with $\mathbf{e}$ as a corresponding eigenvector, and μ is an eigenvalue of the matrix $\mathbf{B}$ for which $\mathbf{e}$ is also a corresponding eigenvector.

(i) Show that $\lambda + \mu$ is an eigenvalue of the matrix $\mathbf{A} + \mathbf{B}$ with $\mathbf{e}$ as a corresponding eigenvector. [2]

The matrix $\mathbf{A}$, given by

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

has $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ as eigenvectors.

(ii) Find the corresponding eigenvalues. [3]

The matrix $\mathbf{B}$ has eigenvalues 4, 5 and 1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ respectively.

(iii) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A} + \mathbf{B})^3 = \mathbf{PDP}^{-1}$. [3]

Q16

9231/12 • October/November 2018 • Question 2

It is given that

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(i) Find the eigenvalue of $\mathbf{A}$ corresponding to the eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. [1]

(ii) Write down the negative eigenvalue of $\mathbf{A}$ and find a corresponding eigenvector. [3]

(iii) Find an eigenvalue and a corresponding eigenvector of the matrix $\mathbf{A} + \mathbf{A}^6$. [2]

Q17

9231/11, 9231/12 • May/June 2019 • Question 9

It is given that $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{A}$, with corresponding eigenvalue λ.

(i) Show that $\mathbf{e}$ is an eigenvector of $\mathbf{A}^2$, with corresponding eigenvalue λ^2. [2]

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = \begin{pmatrix} n & 1 & 3 \\ 0 & 2n & 0 \\ 0 & 0 & 3n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = (\mathbf{A} + n\mathbf{I})^2,$$

where $\mathbf{I}$ is the 3×3 identity matrix and n is a non-zero integer.

(ii) Find, in terms of n, a non-singular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{B} = \mathbf{PDP}^{-1}$. [8]

Q18

9231/13 • May/June 2019 • Question 11

A 3×3 matrix $\mathbf{A}$ has distinct eigenvalues 2, 1, 3, with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

respectively, where b is a positive constant.

(i) Find $\mathbf{A}$ in terms of b. [9]

(ii) Find $\mathbf{A}^{-1}\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$. [2]

(iii) It is given that

$$\mathbf{A}^n\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}^n\begin{pmatrix} -1 \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ b^{-1} \end{pmatrix}.$$

Find the values of n and b. [3]

Q19

9231/11, 9231/12, 9231/13 • October/November 2019 • Question 8

The matrix $\mathbf{M}$ is defined by

$$\mathbf{M} = \begin{pmatrix} 2 & m & 1 \\ 0 & m & 7 \\ 0 & 0 & 1 \end{pmatrix},$$

where $m \neq 0,\ 1,\ 2$.

(i) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$. [7]

(ii) Find $\mathbf{M}^7\mathbf{P}$. [3]

Q20

9231/21, 9231/22 • May/June 2020 • Question 8

(a) Find the values of a for which the system of equations

$$\begin{aligned} 3x + y + z &= 0, \\ ax + 6y - z &= 0, \\ ay - 2z &= 0, \end{aligned}$$

does not have a unique solution. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 6 & -1 \\ 0 & 0 & -2 \end{pmatrix}.$$

(b) Use the characteristic equation of $\mathbf{A}$ to find the inverse of $\mathbf{A}^2$. [4]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [7]

Q21

9231/23 • May/June 2020 • Question 3

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 7 \\ 0 & 6 & 0 \\ 7 & 7 & 5 \end{pmatrix}.$$

(a) Find the eigenvalues of $\mathbf{A}$. [4]

(b) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$. [4]

Q22

9231/21, 9231/23 • October/November 2020 • Question 3

(a) Show that the system of equations

$$x - 2y - 4z = 1,$$

$$x - 2y + kz = 1,$$

$$-x + 2y + 2z = 1,$$

where k is a constant, does not have a unique solution. [2]

(b) Given that $k = -4$, show that the system of equations in part (a) is consistent. Interpret this situation geometrically. [3]

(c) Given instead that $k = -2$, show that the system of equations in part (a) is inconsistent. Interpret this situation geometrically. [2]

(d) For the case where $k \neq -2$ and $k \neq -4$, show that the system of equations in part (a) is inconsistent. Interpret this situation geometrically. [2]

Q23

9231/21, 9231/23 • October/November 2020 • Question 7

The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) State the eigenvalues of $\mathbf{P}$. [1]

(b) Use the characteristic equation of $\mathbf{P}$ to find $\mathbf{P}^{-1}$. [4]

The 3×3 matrix $\mathbf{A}$ has distinct eigenvalues b, -1, 1 with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

respectively.

(c) Find $\mathbf{A}$ in terms of b. [4]

Q24

9231/22 • October/November 2020 • Question 9

It is given that a is a positive constant.

(a) Show that the system of equations

$$ax + (2a+5)y + (a+1)z = 1,$$

$$-4y = 2,$$

$$3y - z = 3,$$

has a unique solution and interpret this situation geometrically. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & 2a+5 & a+1 \\ 0 & -4 & 0 \\ 0 & 3 & -1 \end{pmatrix}.$$

(b) Show that the eigenvalues of $\mathbf{A}$ are a, -1 and -4. [2]

(c) Find a matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{P}\begin{pmatrix} a & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix}\mathbf{P}^{-1}.$$

[5]

(d) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$. [6]

Q25

9231/21, 9231/22 • May/June 2021 • Question 1

(a) Given that a is an integer, show that the system of equations

$$ax + 3y + z = 14, \qquad 2x + y + 3z = 0, \qquad -x + 2y - 5z = 17,$$

has a unique solution and interpret this situation geometrically. [4]

(b) Find the value of a for which $x = 1$, $y = 4$, $z = -2$ is the solution to the system of equations in part (a). [1]

Q26

9231/21, 9231/22 • May/June 2021 • Question 6

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 5 & -\frac{22}{3} & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^2 = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^3$. [4]

Q27

9231/23 • May/June 2021 • Question 8

(a) Find the value of a for which the system of equations

$$13x + 18y - 28z = 0, \qquad -4x - ay + 8z = 0, \qquad 2x + 6y - 5z = 0,$$

does not have a unique solution. [2]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 13 & 18 & -28 \\ -4 & -1 & 8 \\ 2 & 6 & -5 \end{pmatrix}.$$

(b) Find the eigenvalue of $\mathbf{A}$ corresponding to the eigenvector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. [1]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [8]

(d) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$ in terms of $\mathbf{A}$. [2]

Q28

9231/21, 9231/23 • October/November 2021 • Question 2

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 2 & 12 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Use the characteristic equation of $\mathbf{A}$ to show that

$$\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{I},$$

where p and q are integers to be determined. [6]

Q29

9231/22 • October/November 2021 • Question 6

The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 6 & 6 \\ 0 & 2 & 6 \\ 0 & 0 & -3 \end{pmatrix}.$$

(a) Use the characteristic equation of $\mathbf{P}$ to find $\mathbf{P}^{-1}$. [5]

(b) Find the matrix $\mathbf{A}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

[4]

(c) State the eigenvalues and corresponding eigenvectors of $\mathbf{A}^3$. [2]

Q30

9231/21, 9231/22 • May/June 2022 • Question 8

(a) Find the value of a for which the system of equations

$$\begin{aligned} 3x + ay &= 0, \\ 5x - y &= 0, \\ x + 3y + 2z &= 0, \end{aligned}$$

does not have a unique solution. [2]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 5 & -1 & 0 \\ 1 & 3 & 2 \end{pmatrix}.$$

(b) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^2 = \mathbf{PDP}^{-1}$. [7]

(c) Use the characteristic equation of $\mathbf{A}$ to show that

$$(\mathbf{A} + 6\mathbf{I})^2 = \mathbf{A}^4(\mathbf{A} + b\mathbf{I})^2,$$

where b is an integer to be determined. [4]

Q31

9231/23 • May/June 2022 • Question 3

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 6 & -9 & 5 \\ 5 & -8 & 5 \\ 1 & -1 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues of $\mathbf{A}$. [4]

(b) Use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{A}^{-1} = p\mathbf{A}^2 + q\mathbf{I}$, where p and q are constants to be determined. [3]

Q32

9231/21, 9231/23 • October/November 2022 • Question 2

(a) Show that the system of equations

$$x - y + 2z = 4, \qquad x - y - 3z = a, \qquad x - y + 7z = 13,$$

where a is a constant, does not have a unique solution. [2]

(b) Given that $a = -5$, show that the system of equations in part (a) is consistent. Interpret this situation geometrically. [3]

(c) Given instead that $a \neq -5$, show that the system of equations in part (a) is inconsistent. Interpret this situation geometrically. [2]

Q33

9231/21, 9231/23 • October/November 2022 • Question 6

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & -7 \\ 0 & 5 & 7 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^5 = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of $\mathbf{A}$ to show that

$$\mathbf{A}^4 = a\mathbf{A}^2 + b\mathbf{I},$$

where a and b are integers to be determined. [4]

Q34

9231/22 • October/November 2022 • Question 1

(a) Find the set of values of k for which the system of equations

$$x + 2y + 3z = 1, \qquad kx + 4y + 6z = 0, \qquad 7x + 8y + 9z = 3,$$

has a unique solution. [3]

(b) Interpret the situation geometrically in the case where the system of equations does not have a unique solution. [2]

Q35

9231/22 • October/November 2022 • Question 7

(a) It is given that λ is an eigenvalue of the non-singular square matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{e}$.

Show that λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$ for which $\mathbf{e}$ is a corresponding eigenvector. [2]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \\ 15 & -4 & 3 \\ 3 & 0 & 2 \end{pmatrix}.$$

(b) Given that -1 is an eigenvalue of $\mathbf{A}$, find a corresponding eigenvector. [2]

(c) It is also given that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$. Find the corresponding eigenvalues. [2]

(d) Hence find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$. [2]

(e) Use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{A}^{-1} = p\mathbf{A}^2 + q\mathbf{I}$, where p and q are rational numbers to be determined. [4]

Q36

9231/21, 9231/22 • May/June 2023 • Question 1

(a) Show that the system of equations

$$x + 2y + 3z = 1, \qquad 4x + 5y + 6z = 1, \qquad 7x + 8y + 9z = 1,$$

does not have a unique solution. [2]

(b) Show that the system of equations in part (a) is consistent. Interpret this situation geometrically. [3]

Q37

9231/21, 9231/22 • May/June 2023 • Question 5

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 18 & 5 & -11 \\ 8 & 6 & -4 \\ 32 & 10 & -20 \end{pmatrix}.$$

(a) Show that the characteristic equation of $\mathbf{A}$ is $\lambda^3 - 4\lambda^2 - 20\lambda + 48 = 0$ and hence find the eigenvalues of $\mathbf{A}$. [4]

(b) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^5 = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. [6]

Q38

9231/23 • May/June 2023 • Question 8

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & -6a & 2a+2 \\ 0 & 1-a & 0 \\ 0 & 2-a & -1 \end{pmatrix}$$

where a is a constant with $a \neq 0$ and $a \neq 1$.

(a) Show that the equation $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ has a unique solution and interpret this situation geometrically. [3]

(b) Show that the eigenvalues of $\mathbf{A}$ are a, $1-a$ and -1. [2]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^4 = \mathbf{PDP}^{-1}$. [6]

(d) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^4$ in terms of $\mathbf{A}$ and a. [3]

Q39

9231/21, 9231/23 • October/November 2023 • Question 1

Show that the system of equations

$$14x - 4y + 6z = 5,$$

$$x + y + kz = 3,$$

$$-21x + 6y - 9z = 14,$$

where k is a constant, does not have a unique solution and interpret this situation geometrically. [4]

Q40

9231/21, 9231/23 • October/November 2023 • Question 7

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -6 & 2 & 13 \\ 0 & -2 & 5 \\ 0 & 0 & 8 \end{pmatrix}.$$

(a) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$. [4]

Q41

9231/22 • October/November 2023 • Question 6

The matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) State the eigenvalues of $\mathbf{P}$. [1]

(b) Use the characteristic equation of $\mathbf{P}$ to find $\mathbf{P}^{-1}$. [4]

The 3×3 matrix $\mathbf{A}$ has distinct non-zero eigenvalues a, $\frac{1}{2}$, 2 with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

respectively.

(c) Find $\mathbf{A}^{-1}$ in terms of a. [5]

Q42

9231/21, 9231/22 • May/June 2024 • Question 8

(a) Find the set of values of a for which the system of equations

$$6x + ay = 3, \quad 2x - y = 1, \quad x + 5y + 4z = 2$$

has a unique solution. [2]

(b) Show that the system of equations in part (a) is consistent for all values of a. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 6 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 5 & 4 \end{pmatrix}.$$

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(14\mathbf{A} + 24\mathbf{I})^2 = \mathbf{PDP}^{-1}$. [7]

(d) Use the characteristic equation of $\mathbf{A}$ to show that

$$(14\mathbf{A} + 24\mathbf{I})^2 = \mathbf{A}^4(\mathbf{A} + b\mathbf{I})^2,$$

where b is an integer to be determined. [4]

Q43

9231/23 • May/June 2024 • Question 8

The planes Π_1 and Π_2 do not intersect and are both perpendicular to $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$. The line l intersects Π_1 at the point $(1,6,0)$ and intersects Π_2 at the point $(3,-6,0)$.

(a) Find Cartesian equations of Π_1 and Π_2. [3]

(b) Express the vector equation of l in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{a}+\lambda\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors to be determined, and hence show that for points on l, $\dfrac{1}{2}x+\dfrac{1}{12}y=1$ and $z=0$. [2]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{12} & 0 \end{pmatrix}.$$

(c) Show that the characteristic equation of $\mathbf{A}$ is $-\lambda^3+3\lambda^2+\frac{7}{4}\lambda=0$ and hence find the eigenvalues of $\mathbf{A}$. [3]

(d) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^n=\mathbf{PDP}^{-1}$, where n is a positive integer. [6]

Q44

9231/21, 9231/23 • October/November 2024 • Question 1

Find the set of values of k for which the system of equations

$$x + 5y + 6z = 1,$$

$$kx + 2y + 2z = 2,$$

$$-3x + 4y + 8z = 3,$$

has a unique solution and interpret this situation geometrically. [4]

Q45

9231/21, 9231/23 • October/November 2024 • Question 4

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -11 & 1 & 8 \\ 0 & -2 & 0 \\ -16 & 1 & 13 \end{pmatrix}.$$

(a) Show that $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and state the corresponding eigenvalue. [2]

(b) Show that the characteristic equation of $\mathbf{A}$ is $\lambda^3 - 19\lambda - 30 = 0$ and hence find the other eigenvalues of $\mathbf{A}$. [3]

(c) Use the characteristic equation of $\mathbf{A}$ to find $\mathbf{A}^{-1}$. [4]

Q46

9231/22 • October/November 2024 • Question 8

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 7 & 9 \\ 4 & 1 & 7 \end{pmatrix}.$$

(a) Show that the characteristic equation of $\mathbf{A}$ is $\lambda^3 - 12\lambda^2 + 12\lambda + 80 = 0$ and find the eigenvalues of $\mathbf{A}$. [4]

(b) Use the characteristic equation of $\mathbf{A}$ to show that

$$\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{A} + r\mathbf{I},$$

where p, q and r are integers to be determined. [4]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A} - 3\mathbf{I})^4 = \mathbf{PDP}^{-1}$. [6]

Q47

9231/21, 9231/22 • May/June 2025 • Question 8

(a) It is given that λ is an eigenvalue of the non-singular square matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{e}$.

Show that $\mathbf{e}$ is an eigenvector of $\mathbf{A}^3$ with corresponding eigenvalue λ^3. [2]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & -2 & 5 \end{pmatrix}.$$

(b) Show that the eigenvalues of $\mathbf{A}$ are -1, 1 and 5. [2]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} - 2\mathbf{I} = \mathbf{PDP}^{-1}$. [6]

(d) Use the characteristic equation of $\mathbf{A}$ to show that $(\mathbf{A} - 2\mathbf{I})^3 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$ where a, b and c are constants to be determined. [3]

Q48

9231/23 • May/June 2025 • Question 8

(a) Find the values of a for which the system of equations

$$\frac{3}{2}x + 3y + 8z = 1,$$

$$ax + 3y + 4z = 2,$$

$$ay - z = 3,$$

does not have a unique solution. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \frac{3}{2} & 3 & 8 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

(b) Given that $\mathbf{B} = \mathbf{A}^{-1}$, use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{B}^2 = p\mathbf{I} + q\mathbf{A}$, where p and q are constants to be determined. [4]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$. [7]

Q49

9231/24 • May/June 2025 • Question 1

(a) Find the values of k for which the system of equations

$$x + 2y + 3z = 1,$$

$$kx + 5y + 6z = 2,$$

$$7x + 2ky + 9z = 3,$$

does not have a unique solution. [3]

(b) Given that $k = 1$, show that the system of equations in part (a) is consistent. Interpret this situation geometrically. [3]

Q50

9231/24 • May/June 2025 • Question 7

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 7 & 11 \\ 0 & 2 & 5 \\ 0 & 0 & -3 \end{pmatrix}.$$

(a) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^6 = \mathbf{PDP}^{-1}$. [7]

(b) Use the characteristic equation of $\mathbf{A}$ to show that

$$\mathbf{A}^6 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I},$$

where a, b and c are integers to be determined. [4]

Q51

9231/21, 9231/23 • October/November 2025 • Question 1

Find the values of k for which the system of equations

$$x - y + z = 0, \qquad x + ky + 3z = 0, \qquad x + 2y + kz = 0,$$

does not have a unique solution. [3]

Q52

9231/21, 9231/23 • October/November 2025 • Question 6

The matrix $\mathbf{P}$ has non-zero eigenvalues and is given by

$$\mathbf{P} = \begin{pmatrix} a & 1 & 1 \\ 0 & 2a & -1 \\ 0 & 0 & -3a \end{pmatrix}.$$

(a) State, in terms of a, the eigenvalues of $\mathbf{P}$. [1]

(b) (i) Find $\mathbf{P}^2$ in terms of a. [1]

(ii) Use the characteristic equation of $\mathbf{P}$ to find $\mathbf{P}^{-1}$ in terms of a. [3]

The 3×3 matrix $\mathbf{A}$ has eigenvalues $1, 2, 3$ with corresponding eigenvectors

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2a \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -3a \end{pmatrix},$$

respectively.

(c) Find $\mathbf{A}$ in terms of a. [5]

Q53

9231/22 • October/November 2025 • Question 8

(a) Find the values of k for which the system of equations

$$x - y + 2kz = 1, \qquad kx + y + 2z = 2, \qquad 2x - y + z = 3,$$

does not have a unique solution. [3]

(b) Given that $k = -\frac{1}{2}$, show that the system of equations in part (a) is inconsistent. Interpret this situation geometrically. [3]

(c) Given instead that $k = -1$, show that the system of equations in part (a) is also inconsistent. Interpret this situation geometrically. [4]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}.$$

(d) Use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{A}$ where p and q are integers to be determined. [5]

Q54

9231/24 • October/November 2025 • Question 8

(a) Find the set of values of a for which the system of equations

$$5x + ay = 3, \qquad 20x - 5y = 2, \qquad 2x - 3y + z = 1,$$

has a unique solution and interpret this situation geometrically. [3]

(b) Given that $a = -\frac{5}{4}$, show that the system of equations in part (a) is inconsistent and interpret this situation geometrically. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 20 & -5 & 0 \\ 2 & -3 & 1 \end{pmatrix}.$$

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A} + 3\mathbf{I})^2 = \mathbf{PDP}^{-1}$. [7]

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