Further Math with Ali Hashir

# Integration

## Topical Questions (CIE 9231)

Compiled from CIE 9231 Further Mathematics past papers (2015–2025), sorted by topic. Each question is numbered for this booklet, with its original paper code and session shown beneath.

*About this resource:* I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

**What kind of mistakes might you find?** These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

# Q1

9231/11, 9231/12 • May/June 2015 • Question 7

Let $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, dx$, where $n$ is a non-negative integer. Show that

$$I_n = n\left(\tfrac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2}, \quad \text{for } n \geqslant 2.$$

[5]

Find the exact value of $I_4$. [4]

# Q2

9231/13 • May/June 2015 • Question 5

Let $I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin 2n\theta}{\cos\theta}\,d\theta$, where $n$ is a non-negative integer.

(i) Use the identity $\sin P + \sin Q \equiv 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$ to show that

$$I_n + I_{n-1} = \frac{2}{2n-1}, \quad \text{for all positive integers } n.$$

[5]

(ii) Find the exact value of $\int_0^{\frac{1}{2}\pi} \frac{\sin 8\theta}{\cos\theta}\,d\theta$. [4]

# Q3

9231/11, 9231/12 • May/June 2015 • Question 9

The curve $C$ has parametric equations

$$x = 4t + 2t^{\frac{3}{2}}, \quad y = 4t - 2t^{\frac{3}{2}}, \quad \text{for } 0 \leqslant t \leqslant 4.$$

Find the arc length of $C$, giving your answer correct to 3 significant figures. [6]

# Q4

9231/11, 9231/12 • May/June 2016 • Question 5

Let $I_n = \int_0^{\frac{1}{2}\pi} \cos^n x \sin^2 x \, dx$, for $n \geq 0$. By differentiating $\cos^{n-1} x \sin^3 x$ with respect to $x$, prove that

$$(n+2)I_n = (n-1)I_{n-2} \quad \text{for } n \geq 2.$$

[5]

Hence find the exact value of $I_4$. [4]

# Q5

9231/11, 9231/12 • May/June 2016 • Question 11

A curve $C$ has parametric equations

$$x = e^{2t}\cos 2t, \quad y = e^{2t}\sin 2t, \quad \text{for } -\tfrac{1}{2}\pi \le t \le \tfrac{1}{2}\pi.$$

Find the arc length of $C$. [6]

Find the area of the surface generated when $C$ is rotated through $2\pi$ radians about the $x$-axis. [8]

# Q6

9231/13 • May/June 2016 • Question 4

The curve $C$ has equation $y = -\ln(1 - x^2)$ for $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Show that

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1 + x^2}{1 - x^2}\right)^2.$$

[2]

Show further that $\frac{1 + x^2}{1 - x^2}$ may be expressed in the form $\frac{P}{1 + x} + \frac{Q}{1 - x} + R$, where $P$, $Q$ and $R$ are constants to be determined. [2]

Find the exact arc length of $C$. [4]

# Q7

9231/13 • May/June 2016 • Question 6

Let $I_n = \int_0^2 x^n(4 - x^2)^{\frac{1}{2}}\, dx$, for $n \geq 1$. By considering $\frac{d}{dx}\left\{x^n(4 - x^2)^{\frac{3}{2}}\right\}$, show that

$$(n + 3)I_{n+1} = 4nI_{n-1}, \quad \text{where } n \geq 2.$$

[4]

Find the value of $I_1$ and deduce the exact value of $I_3$. [4]

# Q8

9231/11, 9231/12, 9231/13 • October/November 2016 • Question 9

Evaluate $\int_0^{\frac{1}{2}\pi} x \sin x \, dx$. [2]

Given that $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, dx$, prove that, for $n > 1$,

$$I_n = n\left(\tfrac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2}.$$

[4]

By first using the substitution $x = \cos^{-1} u$, find the value of

$$\int_0^1 \left(\cos^{-1} u\right)^3 \, du,$$

giving your answer in an exact form. [5]

# Q9

9231/11, 9231/12 • May/June 2017 • Question 6

Let $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, \mathrm{d}x$.

(i) Prove that, for $n \geqslant 2$,

$$I_n + n(n-1)I_{n-2} = n\left(\tfrac{1}{2}\pi\right)^{n-1}.$$

[4]

(ii) Calculate the exact value of $I_1$ and deduce the exact value of $I_3$. [3]

# Q10

9231/13 • May/June 2017 • Question 5

A curve $C$ has parametric equations

$$x = \tfrac{2}{5}t^{\frac{5}{2}} - 2t^{\frac{1}{2}}, \quad y = \tfrac{4}{3}t^{\frac{3}{2}}, \quad \text{for } 1 \leqslant t \leqslant 4.$$

(i) Find the exact value of the arc length of $C$. [5]

(ii) Find also the exact value of the surface area generated when $C$ is rotated through $2\pi$ radians about the $x$-axis. [3]

# Q11

9231/13 • May/June 2017 • Question 6

Let $I_n$ denote $\int_0^2 (4+x^2)^{-n}\,dx$.

(i) Find $\frac{d}{dx}\left(x(4+x^2)^{-n}\right)$ and hence show that

$$8nI_{n+1} = (2n-1)I_n + 2 \times 8^{-n}.$$

[5]

(ii) Use the result for integrating $\frac{1}{x^2+a^2}$ with respect to $x$, in the List of Formulae (MF10), to find the value of $I_1$ and deduce that

$$I_3 = \tfrac{3}{1024}\pi + \tfrac{1}{128}.$$

[5]

# Q12

9231/11, 9231/12, 9231/13 • October/November 2017 • Question 8

Let $I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \, dx$ for $n > 0$.

(i) Find the value of $I_2$. [2]

(ii) Show that, for $n > 2$,

$$(n-1)I_n = 2^{\frac{1}{2}n-1} + (n-2)I_{n-2}.$$

[5]

(iii) The curve $C$ has equation $y = \sec^3 x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The region $R$ is bounded by $C$, the $x$-axis, the $y$-axis and the line $x = \frac{1}{4}\pi$. Find the volume of revolution generated when $R$ is rotated through $2\pi$ radians about the $x$-axis. [4]

# Q13

9231/11, 9231/12 • May/June 2018 • Question 1

The curve $C$ is defined parametrically by

$$x = e^t - t, \quad y = 4e^{\frac{1}{2}t}.$$

Find the length of the arc of $C$ from the point where $t = 0$ to the point where $t = 3$. [5]

# Q14

9231/11, 9231/12 • May/June 2018 • Question 9

(i) Using the substitution $u = \tan x$, or otherwise, find $\int \sec^2 x \tan^2 x \, dx$. [2]

It is given that, for $n \geqslant 0$,

$$I_n = \int_0^{\frac{1}{4}\pi} \sec^n x \tan^2 x \, dx.$$

(ii) Using the result that $\frac{d}{dx}(\sec x) = \tan x \sec x$, show that, for $n \geqslant 2$,

$$(n+1)I_n = \left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}.$$

[5]

# Q15

9231/11, 9231/13 • October/November 2018 • Question 4

A curve is defined parametrically by

$$x = t - \tfrac{1}{2}\sin 2t \quad \text{and} \quad y = \sin^2 t.$$

The arc of the curve joining the point where $t = 0$ to the point where $t = \pi$ is rotated through one complete revolution about the $x$-axis. The area of the surface generated is denoted by $S$.

(i) Show that

$$S = a\pi \int_0^{\pi} \sin^3 t \, dt,$$

where the constant $a$ is to be found. [5]

(ii) Using the result $\sin 3t = 3\sin t - 4\sin^3 t$, find the exact value of $S$. [3]

# Q16

9231/11, 9231/13 • October/November 2018 • Question 11

The curve $C$ has equation

$$x^2 + 2xy = y^3 - 2.$$

(i) Show that $A(-1, 1)$ is the only point on $C$ with $x$-coordinate equal to $-1$. [2]

For $n \geqslant 1$, let $A_n$ denote the value of $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ at the point $A(-1, 1)$.

(ii) Show that $A_1 = 0$. [3]

(iii) Show that $A_2 = \frac{2}{5}$. [3]

Let $I_n = \int_{-1}^{0} x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\,\mathrm{d}x$.

(iv) Show that for $n \geqslant 2$,

$$I_n = (-1)^{n+1} A_{n-1} - nI_{n-1}.$$

[3]

(v) Deduce the value of $I_3$ in terms of $I_1$. [2]

# Q17

9231/12 • October/November 2018 • Question 11

Let $I_n = \int_1^{\sqrt{2}} (x^2 - 1)^n \, dx$.

(i) Show that, for $n \geqslant 1$,

$$(2n + 1)I_n = \sqrt{2} - 2nI_{n-1}.$$

[5]

(ii) Using the substitution $x = \sec\theta$, show that

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^{2n+1}\theta \sec\theta \, d\theta.$$

[4]

(iii) Deduce the exact value of

$$\int_0^{\frac{1}{4}\pi} \frac{\sin^7\theta}{\cos^8\theta} \, d\theta.$$

[5]

# Q18

9231/11, 9231/12 • May/June 2019 • Question 4

It is given that, for $n \geq 0$,

$$I_n = \int_0^1 x^n e^{x^3} \, dx.$$

(i) Show that $I_2 = \frac{1}{3}(e - 1)$. [2]

(ii) Show that, for $n \geq 3$,

$$3I_n = e - (n-2)I_{n-3}.$$

[3]

(iii) Hence find the exact value of $I_8$. [3]

# Q19

9231/11, 9231/12 • May/June 2019 • Question 5

A curve $C$ is defined parametrically by

$$x = \frac{2}{e^t + e^{-t}} \quad \text{and} \quad y = \frac{e^t - e^{-t}}{e^t + e^{-t}},$$

for $0 \leq t \leq 1$. The area of the surface generated when $C$ is rotated through $2\pi$ radians about the $x$-axis is denoted by $S$.

(i) Show that $S = 4\pi \displaystyle\int_0^1 \frac{e^t - e^{-t}}{(e^t + e^{-t})^2}\,dt$. [5]

(ii) Using the substitution $u = e^t + e^{-t}$, or otherwise, find $S$ in terms of $\pi$ and e. [3]

# Q20

9231/13 • May/June 2019 • Question 10

Let $I_n = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^n x \, dx$, where $n \geqslant 0$.

By considering $\frac{d}{dx}\left(\cot^{n+1} x\right)$, or otherwise, show that

$$I_{n+2} = \frac{1}{n+1} - I_n.$$

[5]

# Q21

9231/11, 9231/12, 9231/13 • October/November 2019 • Question 3

The integral $I_n$, where $n$ is a positive integer, is defined by

$$I_n = \int_{\frac{1}{2}}^{1} x^{-n} \sin \pi x \, dx.$$

(i) Show that

$$n(n+1)I_{n+2} = 2^{n+1}n + \pi - \pi^2 I_n.$$

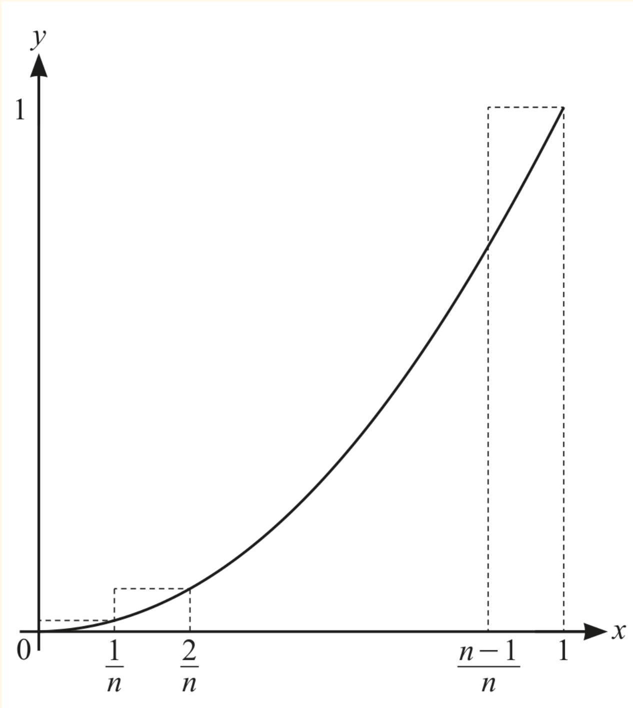
[5]

(ii) Find $I_5$ in terms of $\pi$ and $I_1$. [2]

## Q22

9231/21, 9231/22 • May/June 2020 • Question 4

The diagram shows the curve with equation $y = x^2$ for $0 \leqslant x \leqslant 1$, together with a set of $n$ rectangles of width $\frac{1}{n}$.




(a) By considering the sum of the areas of these rectangles, show that

$$\int_0^1 x^2 \, dx < \frac{2n^2 + 3n + 1}{6n^2}.$$

[4]

(b) Use a similar method to find, in terms of $n$, a lower bound for $\int_0^1 x^2 \, dx$. [4]

## Q23

9231/21, 9231/22 • May/June 2020 • Question 6

The integral $I_n$, where $n$ is an integer, is defined by $I_n = \int_0^{\frac{1}{2}} \left(1 - x^2\right)^{-\frac{1}{2}n} \, \mathrm{d}x$.

(a) Find the exact value of $I_1$. [2]

(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1-x^2\right)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$nI_{n+2} = 2^{n-1}3^{-\frac{1}{2}n} + (n-1)I_n.$$

[5]

(c) Find the exact value of $I_5$ giving the answer in the form $k\sqrt{3}$, where $k$ is a rational number to be determined. [3]

# Q24

9231/23 • May/June 2020 • Question 2

Let $I_n = \int_0^1 (1+3x)^n \mathrm{e}^{-3x}\,\mathrm{d}x$, where $n$ is an integer.

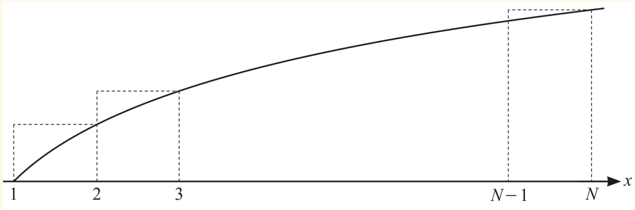
(a) Show that $3I_n = 1 - 4^n \mathrm{e}^{-3} + 3nI_{n-1}$. [3]

(b) Find the exact value of $I_2$. [3]

# Q25

9231/23 • May/June 2020 • Question 4

The diagram shows the curve with equation $y = \ln x$ for $x \geqslant 1$, together with a set of $(N-1)$ rectangles of unit width.




(a) By considering the sum of the areas of these rectangles, show that

$$\ln N! > N \ln N - N + 1.$$

[5]

(b) Use a similar method to find, in terms of $N$, an upper bound for $\ln N!$. [3]

# Q26

9231/23 • May/June 2020 • Question 5

The curve $C$ has parametric equations

$$x = \tfrac{1}{2}t^2 - \ln t, \qquad y = 2t + 1, \qquad \text{for } \tfrac{1}{2} \leqslant t \leqslant 2.$$

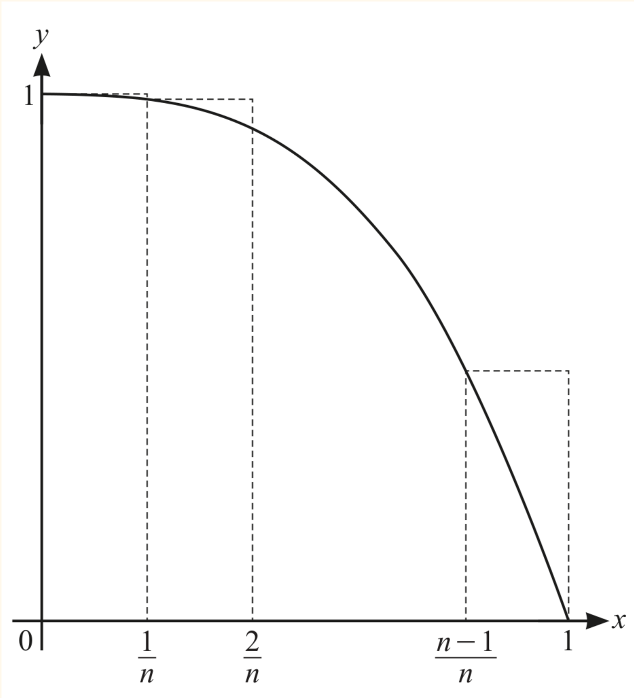
(a) Find the exact length of $C$. [5]

(b) Find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of $t$, simplifying your answer. [4]

## Q27

9231/21, 9231/23 • October/November 2020 • Question 4

The diagram shows the curve with equation $y = 1 - x^3$ for $0 \leqslant x \leqslant 1$, together with a set of $n$ rectangles of width $\frac{1}{n}$.




(a) By considering the sum of the areas of the rectangles, show that

$$\int_0^1 (1 - x^3)\,\mathrm{d}x \leqslant \frac{3n^2 + 2n - 1}{4n^2}.$$

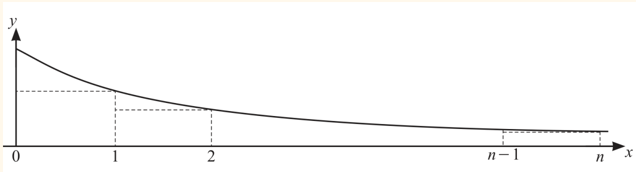
[4]

(b) Use a similar method to find, in terms of $n$, a lower bound for $\int_0^1 (1 - x^3)\,\mathrm{d}x$. [4]

# Q28

9231/22 • October/November 2020 • Question 8

The diagram shows the curve $y = \dfrac{1}{\sqrt{x^2+x+1}}$ for $x \geqslant 0$, together with a set of $n$ rectangles of unit width.




By considering the sum of the areas of these rectangles, show that

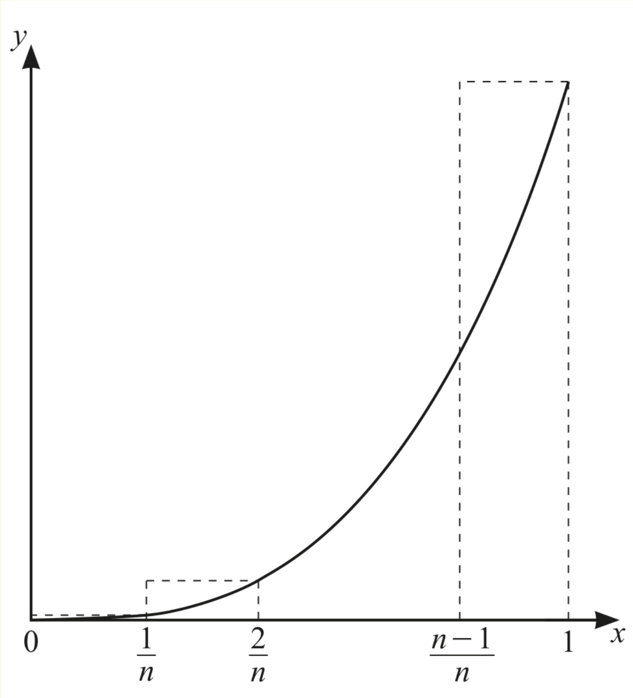
$$\sum_{r=1}^{n} \frac{1}{\sqrt{r^2+r+1}} < \ln\left(\frac{1}{3} + \frac{2}{3}n + \frac{2}{3}\sqrt{n^2+n+1}\right).$$

[10]

## Q29

9231/21, 9231/22 • May/June 2021 • Question 3

The diagram shows the curve with equation $y = x^3$ for $0 \leq x \leq 1$, together with a set of $n$ rectangles of width $\frac{1}{n}$.




(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 x^3 \, dx < U_n$, where

$$U_n = \left(\frac{n+1}{2n}\right)^2.$$

[4]

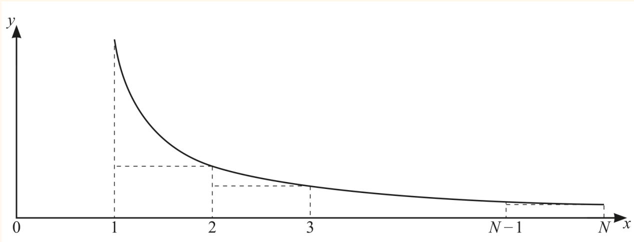
(b) Use a similar method to find, in terms of $n$, a lower bound $L_n$ for $\int_0^1 x^3 \, dx$. [4]

(c) Find the least value of $n$ such that $U_n - L_n < 10^{-3}$. [2]

# Q30

9231/23 • May/June 2021 • Question 3

The diagram shows the curve $y = \dfrac{x}{2x^2 - 1}$ for $x \geq 1$, together with a set of $N - 1$ rectangles of unit width.




(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{N} \frac{r}{2r^2 - 1} < \tfrac{1}{4}\ln(2N^2 - 1) + 1.$$

[7]

(b) Use a similar method to find, in terms of $N$, a lower bound for $\displaystyle\sum_{r=1}^{N} \frac{r}{2r^2 - 1}$. [3]

# Q31

9231/23 • May/June 2021 • Question 7

The integral $I_n$, where $n$ is an integer, is defined by $I_n = \int_0^{\frac{3}{2}} \left(4+x^2\right)^{-\frac{1}{2}n} \, \mathrm{d}x$.

(a) Find the exact value of $I_1$, expressing your answer in logarithmic form. [3]

(b) By considering $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x\left(4+x^2\right)^{-\frac{1}{2}n}\right)$, or otherwise, show that

$$4nI_{n+2} = \tfrac{3}{2}\left(\tfrac{2}{5}\right)^n + (n-1)I_n.$$

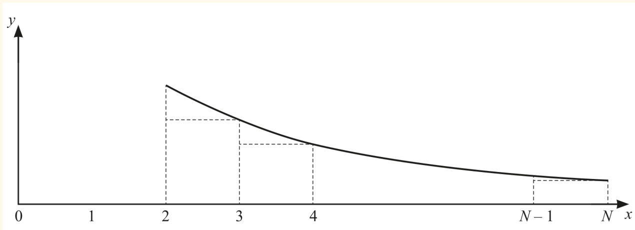
[5]

(c) Find the value of $I_5$. [3]

# Q32

9231/21, 9231/23 • October/November 2021 • Question 4

The diagram shows the curve with equation $y = \dfrac{\ln x}{x^2}$ for $x \geq 2$, together with a set of $(N-2)$ rectangles of unit width.




(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{N} \frac{\ln r}{r^2} < \frac{2 + 3\ln 2}{4} - \frac{1 + \ln N}{N}.$$

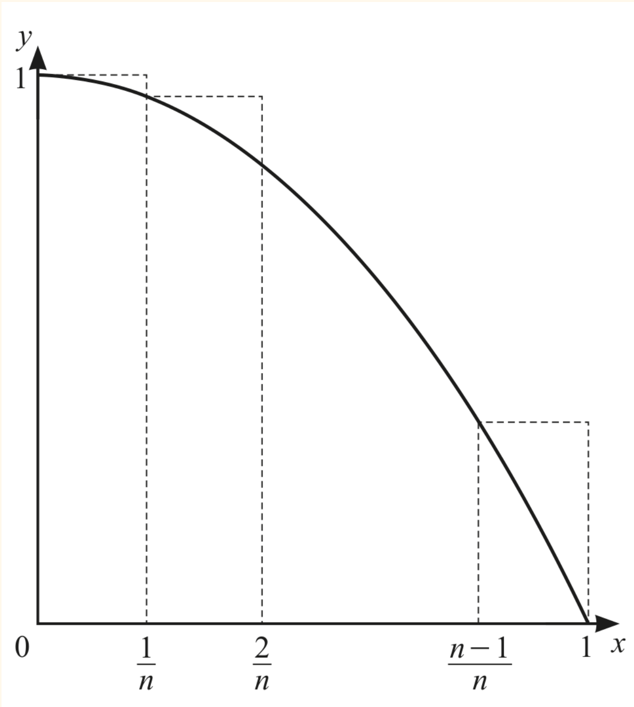
[7]

(b) Use a similar method to find, in terms of $N$, a lower bound for $\displaystyle\sum_{r=1}^{N} \frac{\ln r}{r^2}$. [3]

# Q33

9231/22 • October/November 2021 • Question 3

The diagram shows the curve with equation $y = 1 - x^2$ for $0 \leq x \leq 1$, together with a set of $n$ rectangles of width $\frac{1}{n}$.




(a) By considering the sum of the areas of the rectangles, show that

$$\int_0^1 (1 - x^2)\,dx < \frac{4n^2 + 3n - 1}{6n^2}.$$

[4]

(b) Use a similar method to find, in terms of $n$, a lower bound for $\int_0^1 (1 - x^2)\,dx$. [4]

# Q34

9231/22 • October/November 2021 • Question 5(a)

The curve $C$ has parametric equations

$$x = 3t + 2t^{-1} + at^3, \qquad y = 4t - \tfrac{3}{2}t^{-1} + bt^3, \quad \text{for } 1 \le t \le 2,$$

where $a$ and $b$ are constants.

(a) It is given that $a = \frac{2}{3}$ and $b = -\frac{1}{2}$.

Show that $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{25}{4}(t^2 + t^{-2})^2$ and find the exact length of $C$. [6]

# Q35

9231/22 • October/November 2021 • Question 8(c)(d)

It is given that, for $n \geq 0$, $I_n = \displaystyle\int_0^{\ln 3} \operatorname{sech}^n x \tanh^2 x \, dx$.

(c) Show that, for $n \geq 2$,

$$(n+1)I_n = \left(\frac{4}{5}\right)^3 \left(\frac{3}{5}\right)^{n-2} + (n-2)I_{n-2}.$$

[5]

[You may use the result that $\dfrac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$.]

(d) Find the value of $I_4$. [3]

# Q36

9231/21, 9231/22 • May/June 2022 • Question 1

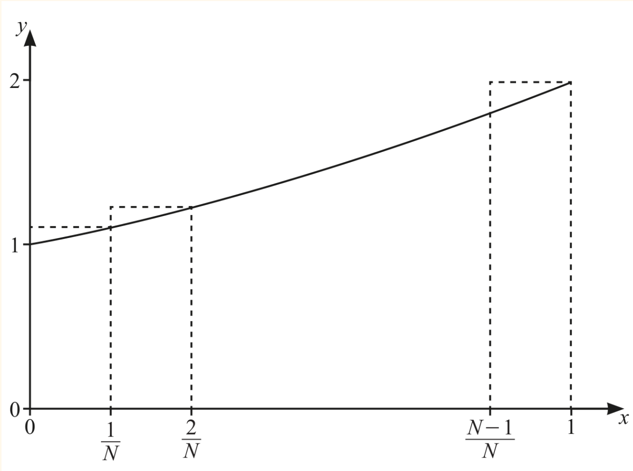
The curve $C$ has polar equation $r = e^{\frac{3}{4}\theta}$ for $0 \leq \theta \leq \alpha$.

Given that the length of $C$ is $s$, find $\alpha$ in terms of $s$. [5]

# Q37

9231/21, 9231/22 • May/June 2022 • Question 4

The diagram shows the curve with equation $y = 2^x$ for $0 \le x \le 1$, together with a set of $N$ rectangles each of width $\frac{1}{N}$.




(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 2^x \, dx < U_N$, where

$$U_N = \frac{2^{\frac{1}{N}}}{N\left(2^{\frac{1}{N}} - 1\right)}.$$

[4]

(b) Use a similar method to find, in terms of $N$, a lower bound $L_N$ for $\int_0^1 2^x \, dx$. [4]

(c) Find the least value of $N$ such that $U_N - L_N < 10^{-4}$. [2]

## Q38

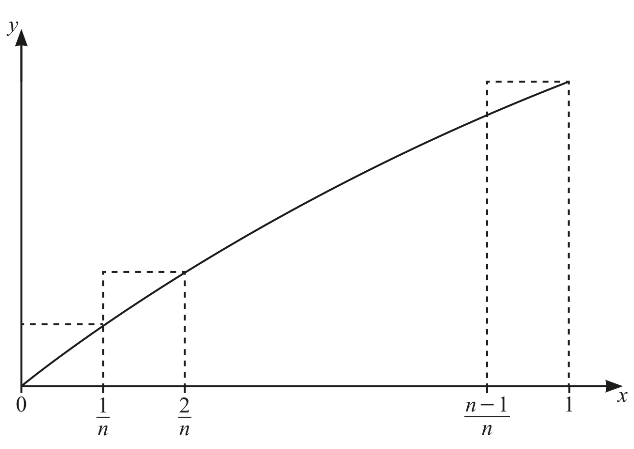
9231/23 • May/June 2022 • Question 2(b)

Find the length of the arc of the curve with equation $y = -\ln\cos x$ from the point where $x = 0$ to the point where $x = \frac{1}{4}\pi$. [4]

## Q39

9231/23 • May/June 2022 • Question 6

The diagram shows the curve with equation $y = \ln(1+x)$ for $0 \le x \le 1$, together with a set of $n$ rectangles each of width $\frac{1}{n}$.




(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \ln(1+x)\,dx < U_n$, where

$$U_n = \frac{1}{n}\ln\frac{(2n)!}{n!} - \ln n.$$

[4]

(b) Use a similar method to find, in terms of $n$, a lower bound $L_n$ for $\int_0^1 \ln(1+x)\,dx$. [4]

(c) By simplifying $U_n - L_n$, show that $\lim_{n\to\infty}(U_n - L_n) = 0$. [2]

# Q40

9231/23 • May/June 2022 • Question 8

(a) Find $\int \sin\theta \cos^n\theta \, d\theta$, where $n \neq -1$. [2]

Let $I_{m,n} = \int_0^{\frac{1}{2}\pi} \sin^m\theta \cos^n\theta \, d\theta$.

(b) Show that, for $m \geq 2$ and $n \geq 0$,

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}.$$

[5]

(c) By considering the binomial expansion of $\left(z + \frac{1}{z}\right)^5$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\cos^5\theta = a\cos 5\theta + b\cos 3\theta + c\cos\theta,$$

where $a$, $b$ and $c$ are constants to be determined. [5]

(d) Using the results given in parts (b) and (c), find the exact value of $I_{2,5}$. [4]

# Q41

9231/21, 9231/23 • October/November 2022 • Question 3

The curve $C$ has parametric equations

$$x = \mathrm{e}^t - \tfrac{1}{3}t^3, \qquad y = 4\mathrm{e}^{\frac{1}{2}t}(t-2), \qquad \text{for } 0 \leq t \leq 2.$$

Find, in terms of e, the length of $C$. [6]

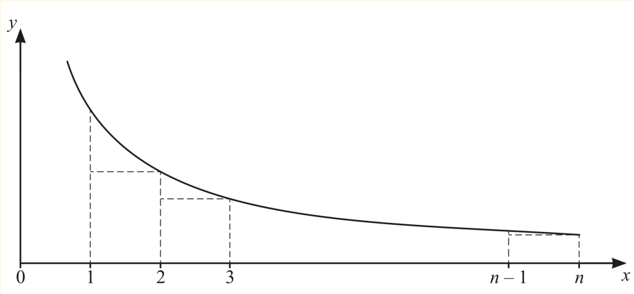
## Q42

9231/22 • October/November 2022 • Question 3(a)

(a) A curve has equation $y = \mathrm{e}^x + \frac{1}{4}\mathrm{e}^{-x}$, for $0 \leq x \leq 1$. Find, in terms of $\pi$ and e, the area of the surface generated when the curve is rotated through $2\pi$ radians about the $x$-axis. [6]

# Q43

9231/22 • October/November 2022 • Question 6




The diagram shows the curve $y = \dfrac{1}{\sqrt{x^2 + 2x}}$ for $x > 0$, together with a set of $(n - 1)$ rectangles of unit width.

By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r^2 + 2r}} < \ln\left(n + 1 + \sqrt{n^2 + 2n}\right) + \tfrac{1}{3}\sqrt{3} - \ln\left(2 + \sqrt{3}\right).$$

[10]

# Q44

9231/21, 9231/22 • May/June 2023 • Question 4

The integral $I_n$ is defined by $I_n = \int_0^1 (1+x^5)^n \, \mathrm{d}x$.

(a) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\big(x(1+x^5)^n\big)$, or otherwise, show that

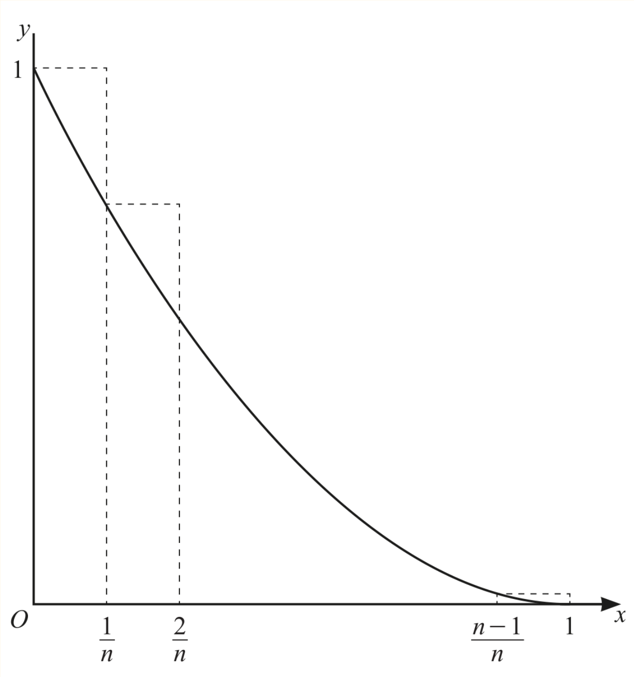
$$(5n+1)I_n = 2^n + 5nI_{n-1}.$$

[5]

(b) Find the exact value of $I_3$. [4]

# Q45

9231/23 • May/June 2023 • Question 6




The diagram shows the curve with equation $y = (1-x)^2$ for $0 \le x \le 1$, together with a set of $n$ rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 (1-x)^2 \, dx < U_n$, where

$$U_n = \frac{2n^2 + 3n + 1}{6n^2}.$$

[5]

(b) Use a similar method to find, in terms of $n$, a lower bound $L_n$ for $\int_0^1 (1-x)^2 \, dx$. [4]

(c) Show that $\lim_{n \to \infty} (U_n - L_n) = 0$. [2]

# Q46

9231/23 • May/June 2023 • Question 7

The integral $I_n$, where $n$ is an integer, is defined by $I_n = \int_0^{\frac{4}{3}} (1+x^2)^{\frac{1}{2}n}\,\mathrm{d}x$.

(a) Find the exact value of $I_{-1}$ giving your answer in the form $\ln a$, where $a$ is an integer to be determined. [2]

(b) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1+x^2)^{\frac{1}{2}n}\right)$, or otherwise, show that

$$(n+1)I_n = nI_{n-2} + \frac{4}{3}\left(\frac{5}{3}\right)^n.$$

[5]

(c) A curve has equation $y = x^2$, for $0 \le x \le \frac{2}{3}$. The arc length of the curve is denoted by $s$.

Use the substitution $u = 2x$ to show that $s = \frac{1}{2}I_1$ and find the exact value of $s$. [4]

# Q47

9231/21, 9231/23 • October/November 2023 • Question 5(a)

The curve $C$ has parametric equations

$$x = \frac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}}, \qquad y = 2t + 5, \qquad \text{for } 0 < t \leqslant 3.$$

(a) Find the exact length of $C$. [5]

## Q48

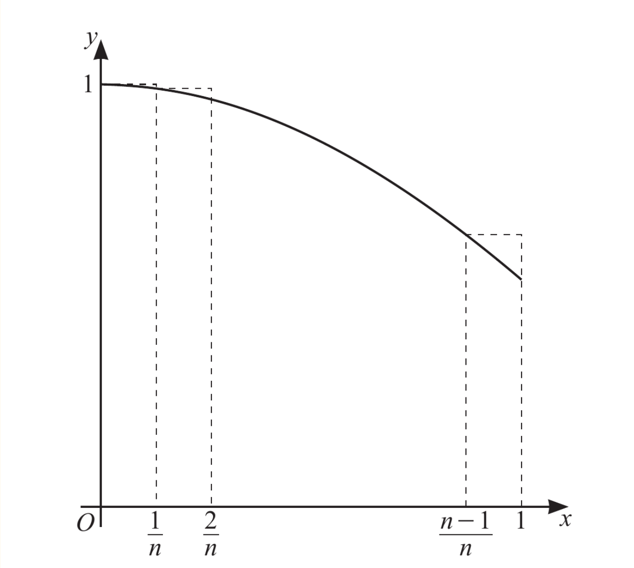
9231/21, 9231/23 • October/November 2023 • Question 8

(a) State the sum of the series $1 + z + z^2 + \cdots + z^{n-1}$, for $z \neq 1$. [1]

(b) By letting $z = \cos\theta + \mathrm{i}\sin\theta$, where $\cos\theta \neq 1$, show that

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta = \tfrac{1}{2}\left(1 - \cos n\theta + \frac{\sin n\theta \sin\theta}{1 - \cos\theta}\right).$$

[7]




The diagram shows the curve with equation $y = \cos x$ for $0 \leqslant x \leqslant 1$, together with a set of $n$ rectangles of width $\dfrac{1}{n}$.

(c) By considering the sum of the areas of these rectangles, show that

$$\int_0^1 \cos x \,\mathrm{d}x < \frac{1}{2n}\left(1 - \cos 1 + \frac{\sin 1 \sin \frac{1}{n}}{1 - \cos \frac{1}{n}}\right).$$

[4]

(d) Use a similar method to find, in terms of $n$, a lower bound for $\displaystyle\int_0^1 \cos x \,\mathrm{d}x$. [3]

# Q49

9231/21, 9231/22 • May/June 2024 • Question 4

It is given that, for $n \geq 0$, $I_n = \displaystyle\int_0^{\ln 3} \operatorname{sech}^n x \, dx$.

(a) Show that, for $n \geq 2$,

$$(n-1)I_n = \left(\frac{3}{5}\right)^{n-2}\left(\frac{4}{5}\right) + (n-2)I_{n-2}.$$

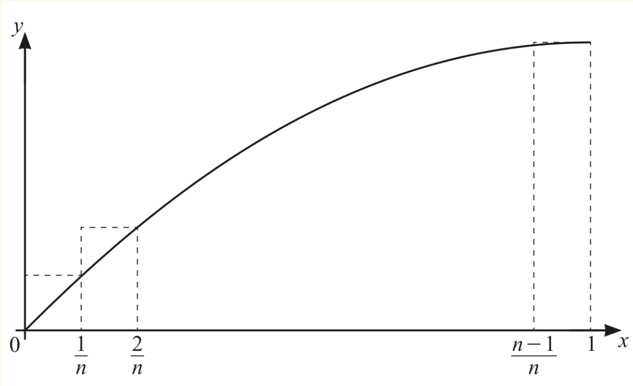
[You may use the result that $\dfrac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$.] [5]

(b) Find the value of $I_4$. [3]

# Q50

9231/21, 9231/22 • May/June 2024 • Question 5

The diagram shows the curve with equation $y = 2x - x^2$ for $0 \leq x \leq 1$, together with a set of $n$ rectangles of width $\frac{1}{n}$.




(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 (2x - x^2)\,\mathrm{d}x < U_n$, where

$$U_n = \left(1 + \frac{1}{n}\right)\left(\frac{2}{3} - \frac{1}{6n}\right).$$

[5]

(b) Use a similar method to find, in terms of $n$, a lower bound $L_n$ for $\int_0^1 (2x - x^2)\,\mathrm{d}x$. [4]

(c) Show that $\lim_{n\to\infty} (U_n - L_n) = 0$. [2]

# Q51

9231/23 • May/June 2024 • Question 1

Find the exact value of $\displaystyle\int_{2}^{7/2} \frac{1}{\sqrt{4x - x^2 - 1}}\,\mathrm{d}x$. [5]

# Q52

9231/23 • May/June 2024 • Question 2

The curve $C$ has parametric equations

$$x = \cosh t, \quad y = \sinh t, \quad \text{for } 0 < t \leq \tfrac{3}{5}.$$

The length of $C$ is denoted by $s$.

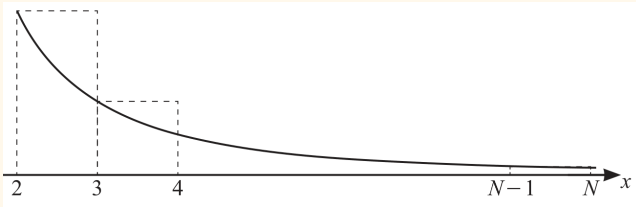
(a) Show that $s = \displaystyle\int_0^{3/5} \sqrt{\cosh 2t}\, \mathrm{d}t$. [4]

(b) By finding the Maclaurin's series for $\sqrt{\cosh 2t}$ up to and including the term in $t^2$, deduce an approximation to $s$. [5]

# Q53

9231/23 • May/June 2024 • Question 4

The diagram shows the curve with equation $y = x^{-2}$ for $2 \le x \le N$ together with a set of $(N-2)$ rectangles of unit width.




(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{N} \frac{1}{r^2} > \frac{3}{2} - \frac{1}{N} + \frac{1}{N^2}.$$

[5]

(b) Use a similar method to find, in terms of $N$, an upper bound for $\sum_{r=1}^{N} \frac{1}{r^2}$. [3]

(c) Deduce lower and upper bounds for $\sum_{r=1}^{\infty} \frac{1}{r^2}$. [2]

# Q54

9231/21, 9231/23 • October/November 2024 • Question 3

A curve has equation $y = e^x$ for $\ln\frac{4}{3} \leq x \leq \ln\frac{12}{5}$. The area of the surface generated when the curve is rotated through $2\pi$ radians about the $x$-axis is denoted by $A$.

(a) Use the substitution $u = e^x$ to show that

$$A = 2\pi \int_{4/3}^{12/5} \sqrt{1+u^2}\, du.$$

[2]

(b) Use the substitution $u = \sinh v$ to show that

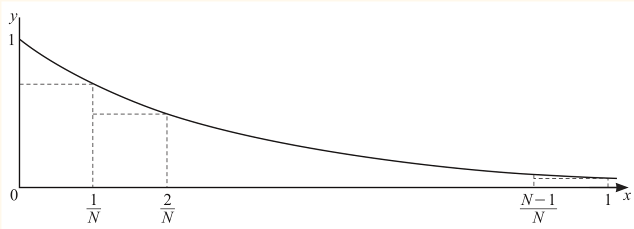
$$A = \pi\left(\frac{904}{225} + \ln\frac{5}{3}\right).$$

[6]

# Q55

9231/21, 9231/23 • October/November 2024 • Question 6

The diagram shows the curve with equation $y = \left(\frac{1}{2}\right)^x$ for $0 \le x \le 1$, together with a set of $N$ rectangles each of width $\frac{1}{N}$.




(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \left(\frac{1}{2}\right)^x dx > L_N$, where

$$L_N = \frac{1}{2N\left(2^{1/N} - 1\right)}.$$

[4]

(b) Use a similar method to find, in terms of $N$, an upper bound $U_N$ for $\int_0^1 \left(\frac{1}{2}\right)^x dx$. [4]

(c) Find the least value of $N$ such that $U_N - L_N \le 10^{-3}$. [2]

(d) Given that $\int_0^1 \left(\frac{1}{2}\right)^x dx = \frac{1}{2\ln 2}$, use the value of $N$ found in part (c) to find upper and lower bounds for $\ln 2$. [4]

# Q56

9231/22 • October/November 2024 • Question 1

Find the value of $\int_6^7 \frac{1}{\sqrt{(x-5)^2-1}}\,dx$, giving your answer in the form $\ln(a+\sqrt{b})$, where $a$ and $b$ are integers to be determined. [4]

# Q57

9231/22 • October/November 2024 • Question 3

The curve $C$ has parametric equations

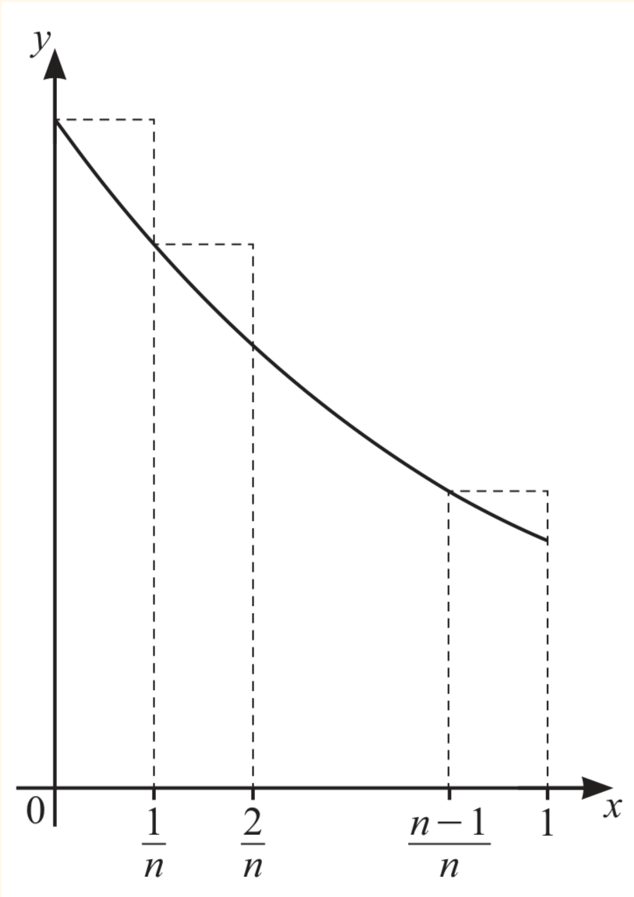
$$x = \frac{1}{2}e^{2t} - \frac{1}{3}t^3 - \frac{1}{2}, \qquad y = 2e^t(t-1), \qquad \text{for } 0 \leq t \leq 1.$$

Find the exact length of $C$. [7]

## Q58

9231/22 • October/November 2024 • Question 6

The diagram shows the curve with equation $y = e^{1-x}$ for $0 \leq x \leq 1$, together with a set of $n$ rectangles of width $\dfrac{1}{n}$.




(a) By considering the sum of the areas of these rectangles, show that $\displaystyle\int_0^1 e^{1-x}dx < U_n$, where

$$U_n = \frac{e-1}{n\left(1-e^{-1/n}\right)}.$$

[4]

(b) Use a similar method to find, in terms of $n$, a lower bound $L_n$ for $\displaystyle\int_0^1 e^{1-x}dx$. [4]

(c) Show that $\displaystyle\lim_{n\to\infty}(U_n - L_n) = 0$. [2]

(d) Use the Maclaurin's series for $e^x$ given in the list of formulae (MF19) to find the first three terms of the series expansion of $z\left(1-e^{-1/z}\right)$, in ascending powers of $\dfrac{1}{z}$, and deduce the value of $\displaystyle\lim_{n\to\infty}(U_n)$. [3]

# Q59

9231/21, 9231/22 • May/June 2025 • Question 2

Let $I_n = \int_0^1 (1-x)^n \sinh x \, \mathrm{d}x$, where $n$ is a non-negative integer.

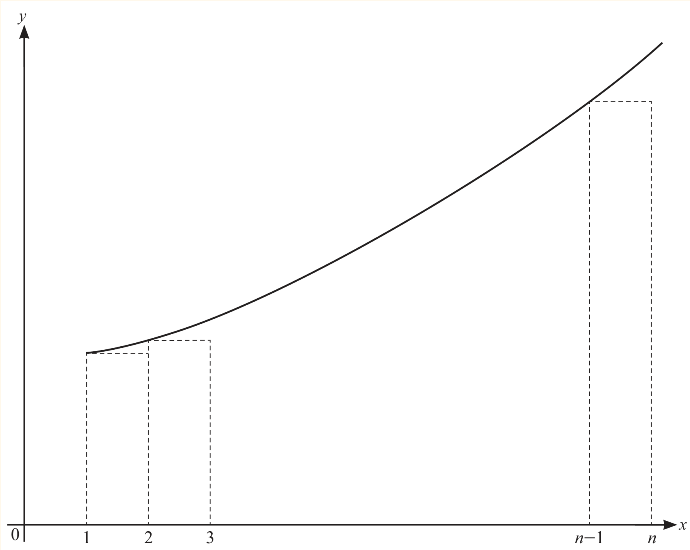
(a) Show that, for $n \geqslant 2$, $I_n = -1 + n(n-1)I_{n-2}$. [4]

(b) Find the exact value of $I_2$. [3]

# Q60

9231/21, 9231/22 • May/June 2025 • Question 4

The diagram shows the curve with equation $y = \frac{1}{\sqrt{x}}e^{\sqrt{x}}$ for $x \geqslant 1$, together with a set of $n-1$ rectangles of unit width.




(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}}e^{\sqrt{r}} < \left(2 + \frac{1}{\sqrt{n}}\right)e^{\sqrt{n}} - 2e.$$

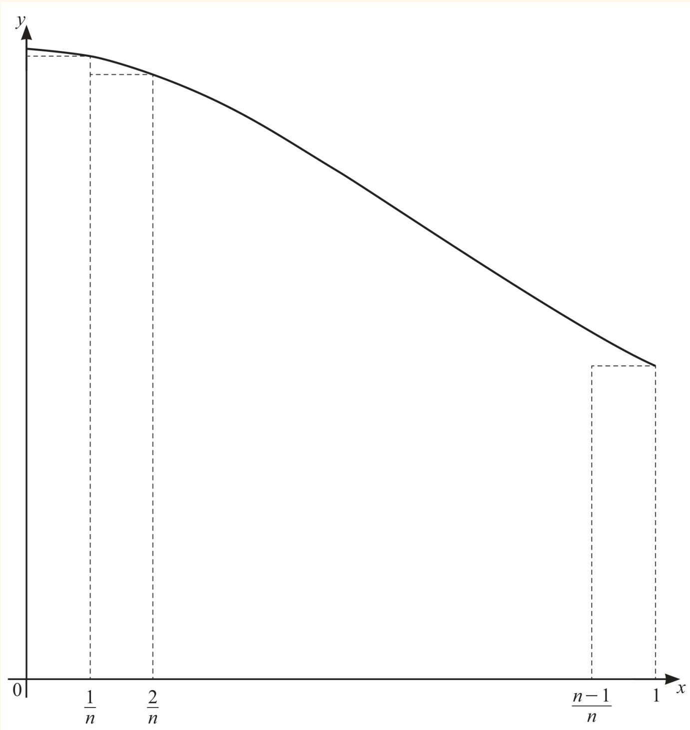
[5]

(b) Use a similar method to find, in terms of $n$, a lower bound for $\sum_{r=1}^{n} \frac{1}{\sqrt{r}}e^{\sqrt{r}}$. [4]

# Q61

9231/23 • May/June 2025 • Question 6

The diagram shows the curve with equation $y = \dfrac{1}{x^2+1}$ for $0 \leqslant x \leqslant 1$, together with a set of $n$ rectangles of width $\dfrac{1}{n}$.




(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{n}{n^2+r^2} < \frac{1}{4}\pi.$$

[5]

(b) Use a similar method to find a lower bound for $\sum_{r=1}^{n} \dfrac{n}{n^2+r^2}$. Give your answer in terms of $n$ and $\pi$. [4]

(c) Deduce the exact value of $\lim\limits_{n\to\infty} \sum_{r=1}^{n} \dfrac{n}{n^2+r^2}$. [1]

# Q62

9231/24 • May/June 2025 • Question 2

Find the exact value of $\int_1^{5/2} \frac{1}{\sqrt{x^2 - 2x + 5}}\,\mathrm{d}x$, giving your answer in logarithmic form. [6]

# Q63

9231/21, 9231/23 • October/November 2025 • Question 3

The integral $I_n$ is defined by $I_n = \int_0^1 \left(1+x^2\right)^n \mathrm{d}x$.

(a) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1+x^2)^n\right)$, or otherwise, show that

$$(2n+1)I_n = 2^n + 2nI_{n-1}.$$

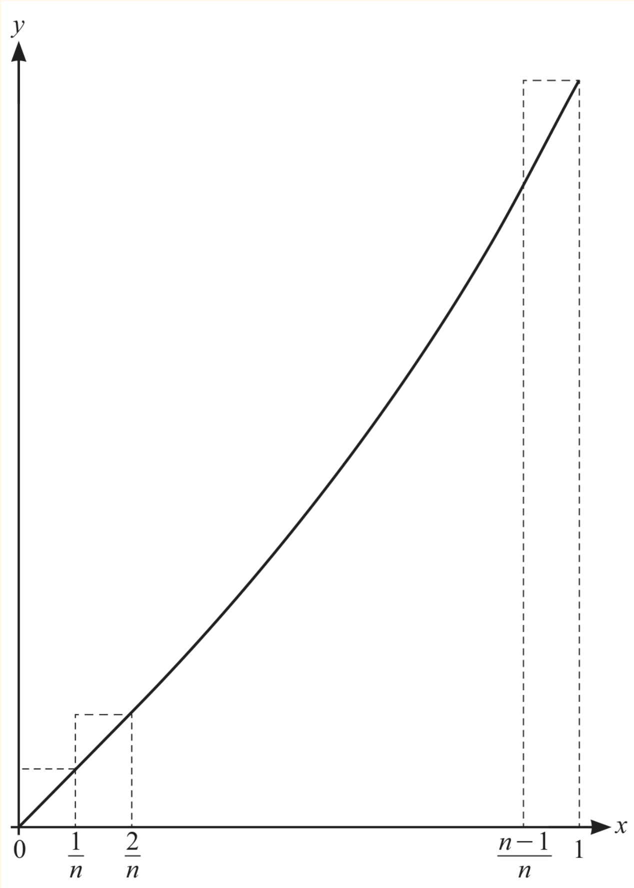
[5]

(b) Find the exact value of $I_{-2}$. [4]

# Q64

9231/21, 9231/23 • October/November 2025 • Question 5

The diagram shows the curve with equation $y = \frac{1}{3}x^3 + x$ for $0 \le x \le 1$, together with a set of $n$ rectangles of width $\frac{1}{n}$.




(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \left(\frac{1}{3}x^3 + x\right) \mathrm{d}x < U_n$, where

$$U_n = \frac{1}{12}\left(1 + \frac{1}{n}\right)\left(7 + \frac{1}{n}\right).$$

[5]

(b) Use a similar method to find, in terms of $n$, a lower bound $L_n$ for $\int_0^1 \left(\frac{1}{3}x^3 + x\right) \mathrm{d}x$. [4]

(c) Show that $\lim_{n\to\infty} (U_n - L_n) = 0$. [2]

# Q65

9231/22 • October/November 2025 • Question 1

The curve $C$ has polar equation $r = e^{\frac{4}{3}\theta}$ for $0 \le \theta \le \pi$.

Find the length of $C$. Give your answer in an exact form. [4]

# Q66

9231/22 • October/November 2025 • Question 4(b)

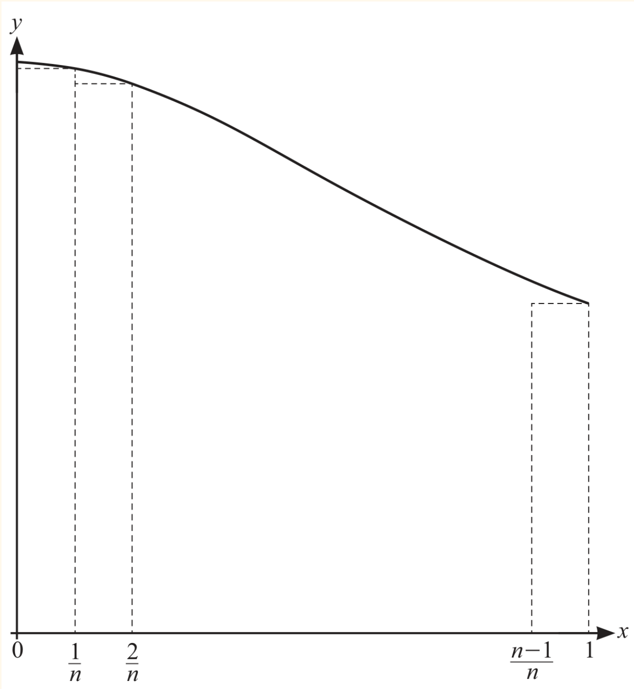
Find the exact value of $\int_0^{\frac{1}{2}\pi} \cos^6 2x \, \mathrm{d}x$. [3]

# Q67

9231/22 • October/November 2025 • Question 6

(a) Use the substitution $x = \frac{1}{2}\sqrt{2}\sinh u$ to find $\displaystyle\int \frac{1}{\sqrt{2x^2+1}}\,\mathrm{d}x$. [3]

The diagram shows the curve with equation $y = \dfrac{1}{\sqrt{2x^2+1}}$ for $0 \le x \le 1$, together with a set of $n$ rectangles of width $\frac{1}{n}$.




(b) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}} < \frac{1}{2}\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right).$$

[5]

(c) Use a similar method to find, in terms of $n$, a lower bound for $\displaystyle\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}}$. [4]

(d) Deduce the exact value of $\displaystyle\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}}$. [1]

# Q68

9231/24 • October/November 2025 • Question 1

It is given that $I_n = \int_0^1 x^n \sinh x \, dx$, where $n$ is a non-negative integer.

(a) Show that, for $n \geq 2$,

$$I_n = \cosh 1 - n \sinh 1 + n(n-1)I_{n-2}.$$

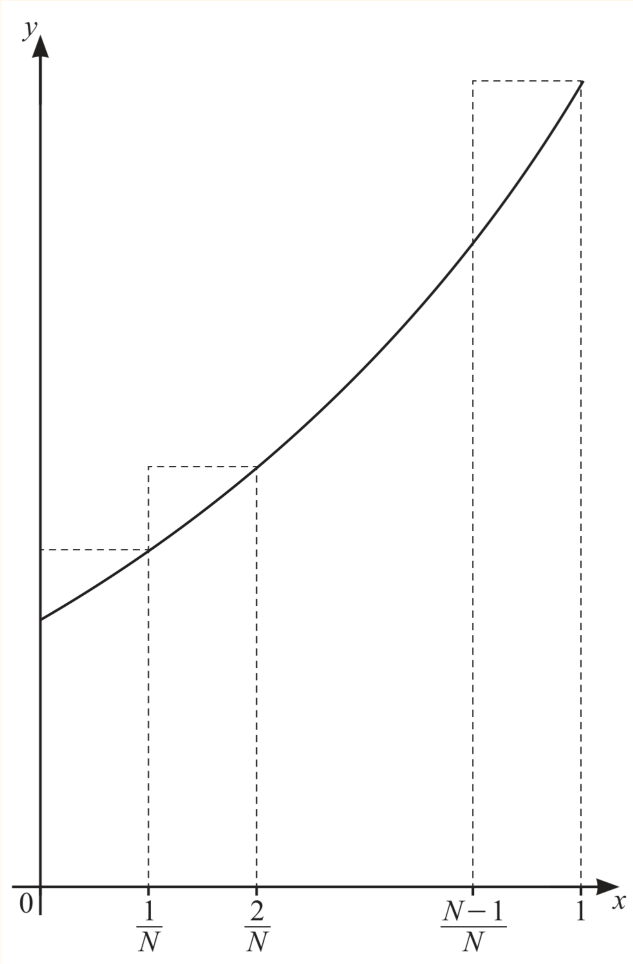
[3]

(b) Find the exact value of $I_2$. [2]

## Q69

9231/24 • October/November 2025 • Question 4

The diagram shows the curve with equation $y = \frac{1}{2}(3^x)$ for $0 \leq x \leq 1$, together with a set of $N$ rectangles each of width $\frac{1}{N}$.




(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \frac{1}{2}(3^x)\,\mathrm{d}x < U_N$, where

$$U_N = \frac{3^{\frac{1}{N}}}{N\left(3^{\frac{1}{N}} - 1\right)}.$$

[4]

(b) Use a similar method to find, in terms of $N$, a lower bound $L_N$ for $\int_0^1 \frac{1}{2}(3^x)\,\mathrm{d}x$. [4]

(c) By simplifying $U_N - L_N$, show that $\lim_{N\to\infty}(U_N - L_N) = 0$. [2]

# Q70

9231/24 • October/November 2025 • Question 7(a)

The curve $C$ has parametric equations

$$x = 8\ln\left(\tan\tfrac{1}{2}t\right) - 3\cot t - 3t, \qquad y = 6\ln\left(\tan\tfrac{1}{2}t\right) + 4\cot t + 4t, \qquad \text{for } \tfrac{1}{6}\pi \le t \le \tfrac{1}{3}\pi.$$

(a) (i) Show that $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 8\operatorname{cosec} t + 3\cot^2 t$. [1]

(ii) Find $\dfrac{\mathrm{d}y}{\mathrm{d}t}$. [1]

(iii) Find the exact value of the length of $C$. [6]

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*Written by Ali Hashir.*



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