Further Math with Ali Hashir

Integration

Topical Mark Schemes (CIE 9231)

Mark schemes corresponding to the topical questions booklet for this topic, compiled from official CIE 9231 marking schemes (2015–2025). Question numbers match the questions booklet.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Contents

Click a question number below to jump straight to its mark scheme. Every box is a live link.

For the best hyperlink experience, open this PDF in a dedicated reader – such as Adobe Acrobat, Apple Preview, or your browser's built-in PDF viewer – rather than a basic in-app or embedded preview, where links sometimes don't work.

Q1

9231/11, 9231/12 • May/June 2015 • Question 7

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Mark scheme.

$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx = [-x^n \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1} \cos x \, dx$ (LNR)	**M1A1**
$= 0 + [nx^{n-1} \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2} \sin x \, dx$ (LR)	**M1A1**
$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$ (AG)	**A1**
$I_0 = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$	**B1**
$I_2 = \pi - 2$, using reduction formula.	**M1A1**
$I_4 = 4 \times \left(\frac{\pi}{2}\right)^3 - 12(\pi - 2) = \frac{1}{2}\pi^3 - 12\pi + 24$	**A1**
(If I_2 found by integration M1A1 (if correct) then M1A1 for I_4 from reduction formula.)	
	Total: 9

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Q2

9231/13 • May/June 2015 • Question 5

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Mark scheme.

(i) $I_n + I_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin(2n\theta) + \sin(2n-2)\theta}{\cos\theta}\,d\theta = \int_0^{\frac{\pi}{2}} \frac{2\sin(2n-1)\theta\cos\theta}{\cos\theta}\,d\theta$ (LR)	**M1A1**
$= \int_0^{\frac{\pi}{2}} 2\sin(2n-1)\theta\,d\theta = \left[-\frac{2\cos(2n-1)\theta}{(2n-1)}\right]_0^{\frac{\pi}{2}}$ (LR)	**dM1A1**
$= [0] - \left[-\frac{2}{(2n-1)}\right] = \frac{2}{2n-1}$ (AG)	**A1**
(ii) $I_1 = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\cos\theta}\,d\theta = \int_0^{\frac{\pi}{2}} 2\sin\theta\,d\theta = [-2\cos\theta]_0^{\frac{\pi}{2}} = [0] - [-2] = 2$	**B1**
$I_2 = \frac{2}{3} - 2 = -\frac{4}{3}$	**M1A1**
$\Rightarrow I_3 = \frac{2}{5} + \frac{4}{3} = \frac{26}{15} \Rightarrow I_4 = \frac{2}{7} - \frac{26}{15} = -\frac{152}{105}$	**A1**
	Total: 9

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Q3

9231/11, 9231/12 • May/June 2015 • Question 9

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Mark scheme.

$\dot{x} = 4 + 3t^{\frac{1}{2}}$, $\dot{y} = 4 - 3t^{\frac{1}{2}}$	**B1**
$\dot{x}^2 + \dot{y}^2 = 32 + 18t$	**B1**
$s = \int_0^4 (32 + 18t)^{\frac{1}{2}}\, dt = \left[\frac{1}{27}(32 + 18t)^{\frac{3}{2}}\right]_0^4$	**M1A1**
$= \frac{1}{27}\left(104^{\frac{3}{2}} - 32^{\frac{3}{2}}\right) = 32.6$ (CAO)	**M1A1**
	Total: 6

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Q4

9231/11, 9231/12 • May/June 2016 • Question 5

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Mark scheme.

$\frac{d}{dx}\left(\cos^{n-1}x\sin^3 x\right) = -(n-1)\cos^{n-2}x\sin^4 x + 3\cos^n x\sin^2 x$	**M1A1**
$\Rightarrow \left[\cos^{n-1}x\sin^3 x\right]_0^{\frac{1}{2}\pi} = -\int_0^{\frac{1}{2}\pi}(n-1)\cos^{n-2}x\sin^2 x(1-\cos^2 x)\,dx + 3I_n$	**M1**
$\Rightarrow 0 = -(n-1)I_{n-2} + (n-1)I_n + 3I_n$	**M1**
$\Rightarrow (n+2)I_n = (n-1)I_{n-2}$ (AG)	**A1**
$I_0 = \int_0^{\frac{1}{2}\pi}\sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{1}{2}\pi}(1-\cos 2x)\,dx$	**M1**
$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\frac{1}{2}\pi} = \frac{\pi}{4}$	**A1**
$I_2 = \frac{1}{4}\times\frac{\pi}{4} \quad I_4 = \frac{1}{2}\times\frac{1}{4}\times\frac{\pi}{4} = \frac{\pi}{32}$	**M1A1**
	Total: 9

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Q5

9231/11, 9231/12 • May/June 2016 • Question 11

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Mark scheme.

$\dot{x} = 2e^{2t}\cos 2t - 2e^{2t}\sin 2t,\ \dot{y} = 2e^{2t}\cos 2t + 2e^{2t}\sin 2t$	**B1**
$\dot{x}^2+\dot{y}^2 = 4e^{4t}\left(\cos^2 2t - 2\cos 2t\sin 2t + \sin^2 2t + \cos^2 2t + 2\cos 2t\sin 2t + \sin^2 2t\right)$	**M1**
$= 8e^{4t}$	**A1**
$s = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 2\sqrt{2}e^{2t}\,dt$	**M1**
$= \sqrt{2}\left[e^{2t}\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} = \sqrt{2}\left(e^{\pi} - e^{-\pi}\right)$ or $2\sqrt{2}\sinh\pi$ or 32.7	**M1A1**
$S = 2\pi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{2t}\sin 2t.2\sqrt{2}e^{2t}\,dt = 4\sqrt{2}\pi\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{4t}\sin 2t\,dt$	**M1**
Let $I = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} e^{4t}\sin 2t\,dt = \left[-e^{4t}\frac{\cos 2t}{2}\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} + \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 2e^{4t}\cos 2t\,dt$	**M1A1**
$= \left[\frac{e^{2\pi}}{2}\right] - \left[\frac{e^{-2\pi}}{2}\right] + 2\left\{\left[e^{4t}\frac{\sin 2t}{2}\right]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} - \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} 2e^{4t}\sin 2t\right\}$	**A1A1**
$= \frac{e^{2\pi} - e^{-2\pi}}{2} + 0 - 4I$	**M1**
$\Rightarrow I = \frac{e^{2\pi} - e^{-2\pi}}{10}$	**A1**
$\Rightarrow S = 4\sqrt{2}\pi.\frac{e^{2\pi} - e^{-2\pi}}{10} = \frac{2\sqrt{2}\pi}{5}\left(e^{2\pi} - e^{-2\pi}\right)$ or $\frac{4\sqrt{2}\pi}{5}\sinh 2\pi$ or 952 (3sf)	**A1**
	Total: 14

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Q6

9231/13 • May/June 2016 • Question 4

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Mark scheme.

$\frac{dy}{dx} = \frac{2x}{1-x^2}$	**B1**
$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4x^2}{(1-x^2)^2} = \frac{1-2x^2+x^4+4x^2}{(1-x^2)^2} = \frac{(1+x^2)^2}{(1-x^2)^2}$ (AG)	**B1**
$\frac{1+x^2}{1-x^2} \equiv \frac{1}{1+x} + \frac{1}{1-x} - 1$	**M1A1**
$s = 2\int_0^{\frac{1}{2}} \left(\frac{1}{1+x} + \frac{1}{1-x} - 1\right) dx$ (oe)	**M1**
$= 2\left[\ln\left(\frac{1+x}{1-x}\right) - x\right]_0^{\frac{1}{2}}$	**M1A1**
$= 2\ln 3 - 1$ (AEF)	**A1**
	Total: 8

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Q7

9231/13 • May/June 2016 • Question 6

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Mark scheme.

$\frac{d}{dx}\left\{x^n(4-x^2)^{\frac{3}{2}}\right\} = -3x^{n+1}(4-x^2)^{\frac{1}{2}} + nx^{n-1}(4-x^2)^{\frac{3}{2}}$	**B1**
$-3x^{n+1}(4-x^2)^{\frac{1}{2}} + nx^{n-1}(4-x^2)^{\frac{1}{2}}(4-x^2)$	**M1**
$\Rightarrow \left[x^n(4-x^2)^{\frac{3}{2}}\right]_0^2 = 0 = -3I_{n+1} + 4nI_{n-1} - nI_{n+1}$	**M1**
$\Rightarrow (n+3)I_{n+1} = 4nI_{n-1}$ (AG)	**A1**
$I_1 = \left[-\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right]_0^2 = \frac{8}{3}$	**M1A1**
$5I_3 = 8I_1 \Rightarrow I_3 = \frac{64}{15}$	**M1A1**
	Total: 8

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Q8

9231/11, 9231/12, 9231/13 • October/November 2016 • Question 9

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Mark scheme.

$\int_0^{\frac{\pi}{2}} x\sin x\,\mathrm{d}x = [-x\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x = 0 + [\sin x]_0^{\frac{\pi}{2}} = 1$	**M1A1**
$I_n = [-x^n\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1}\cos x\,\mathrm{d}x$	**M1A1**
$= \left[-x^n\cos x + nx^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2}\sin x\,\mathrm{d}x$	**A1**
$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$	**A1**
$x = \cos^{-1} u \Rightarrow u = \cos x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x;\ u = 0, 1 \Rightarrow x = \pi/2, 0$	**B1B1**
$\int_0^1 \left(\left[\cos^{-1} u\right]^3\right)\mathrm{d}u = -\int_{\pi/2}^0 x^3\sin x\,\mathrm{d}x = \int_0^{\pi/2} x^3\sin x\,\mathrm{d}x = I_3$	**B1**
$I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3\times 2\times I_1 = \frac{3}{4}\pi^2 - 6$ (OE)	**M1A1**
	Total: 11

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Q9

9231/11, 9231/12 • May/June 2017 • Question 6

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Mark scheme.

$I_n = [-x^n \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} nx^{n-1} \cos x \, dx$	**M1A1**
$= 0 + \left[nx^{n-1} \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2} \sin x \, dx$	**A1**
$= n\left(\frac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2} \Rightarrow I_n + n(n-1)I_{n-2} = n\left(\frac{1}{2}\pi\right)^{n-1}$ (AG)	**A1**
$I_1 = \int_0^{\frac{\pi}{2}} x \sin x \, dx = [-x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$	**M1**
$= [\sin x]_0^{\frac{\pi}{2}} = 1$	**A1**
$n = 3 \Rightarrow I_3 = 3\left(\frac{\pi}{2}\right)^2 - 3 \times 2 \times 1 = \frac{3}{4}\pi^2 - 6$	**B1 FT**
	Total: 7

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Q10

9231/13 • May/June 2017 • Question 5

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Mark scheme.

$\dot{x}^2 + \dot{y}^2 = \left(t^{\frac{3}{2}} - t^{-\frac{1}{2}}\right)^2 + 4t = \ldots = \left(t^{\frac{3}{2}} + t^{-\frac{1}{2}}\right)^2$	**M1A1**
$s = \int_1^4 \left(t^{\frac{3}{2}} + t^{-\frac{1}{2}}\right) \mathrm{d}t$	**M1**
$= \left[\frac{2}{5}t^{\frac{5}{2}} + 2t^{\frac{1}{2}}\right]_1^4$	**M1**
$= \left[\frac{64}{5} + 4\right] - \left[\frac{2}{5} + 2\right] = \frac{72}{5}$	**A1**
$S = 2\pi \int_1^4 \frac{4}{3}t^{\frac{3}{2}} \left(t^{\frac{3}{2}} + t^{-\frac{1}{2}}\right) \mathrm{d}t = \frac{8}{3}\pi \int_1^4 \left(t^3 + t\right) \mathrm{d}t$	***M1**
$= \left(\frac{8}{3}\pi\right) \left[\frac{1}{4}t^4 + \frac{1}{2}t^2\right]_1^4$	**DM1**
$= \frac{8}{3}\pi \left\{[64 + 8] - \left[\frac{1}{4} + \frac{1}{2}\right]\right\} = 190\pi$	**A1**
	Total: 8

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Q11

9231/13 • May/June 2017 • Question 6

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Mark scheme.

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{x\left(4+x^2\right)^{-n}\right\} = \left(4+x^2\right)^{-n} - 2nx^2\left(4+x^2\right)^{-n-1}$	**M1A1**
Integrate w.r.t. x: $\left[x\left(4+x^2\right)^{-n}\right]_0^2 = I_n - 2n\displaystyle\int_0^2 \left(4+x^2-4\right)\left(4+x^2\right)^{-n-1}\mathrm{d}x$	**M1**
	M1
$\Rightarrow 2.8^{-n} = I_n - 2nI_n + 8nI_{n+1} \Rightarrow 8nI_{n+1} = (2n-1)I_n + 2.8^{-n}$	**A1**
$I_1 = \displaystyle\int_0^2 \frac{1}{4+x^2}\mathrm{d}x = \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$	**M1A1**
$8I_2 = \dfrac{\pi}{8} + \dfrac{1}{4} \Rightarrow I_2 = \dfrac{\pi}{64} + \dfrac{1}{32}$	**M1A1FT**
$16I_3 = \dfrac{3\pi}{64} + \dfrac{3}{32} + \dfrac{1}{32} \Rightarrow I_3 = \dfrac{3\pi}{1024} + \dfrac{1}{128}$	**A1**
	Total: 10

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Q12

9231/11, 9231/12, 9231/13 • October/November 2017 • Question 8

↑ Back to contents

Mark scheme.

$I_2 = \int_0^{\frac{1}{4}\pi} \sec^2 x \, dx = [\tan x]_0^{\frac{1}{4}\pi} = 1$	**M1A1**
$I_n = \int_0^{\frac{1}{4}\pi} \sec^{n-2} x \cdot \sec^2 x \, dx$	**M1**
$= \left[\sec^{n-2} x \tan x\right]_0^{\frac{1}{4}\pi} - \int_0^{\frac{1}{4}\pi} (n-2)\sec^{n-3} x(\sec x \tan x) \tan x \, dx$	**M1A1**
$= \left[\sec^{n-2} x \tan x\right]_0^{\frac{1}{4}\pi} - (n-2)\int_0^{\frac{1}{4}\pi} \sec^{n-2} x \left(\sec^2 x - 1\right) dx$	**M1A1**
$\Rightarrow (n-1)I_n = 2^{\frac{1}{2}n-1} + (n-2)I_{n-2}$ (AG)	
Volume of revolution $= \pi \int y^2 \, dx = \pi \int_0^{\frac{1}{4}\pi} \sec^6 x \, dx$	**M1**
$3I_4 = 2 + 2 \times 1 \Rightarrow I_4 = \frac{4}{3}$	**M1**
$5I_6 = 4 + 4 \times \frac{4}{3} \Rightarrow I_6 = \frac{28}{15}$	**M1**
Volume of revolution $= \frac{28\pi}{15}$	**A1**
	Total: 11

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Q13

9231/11, 9231/12 • May/June 2018 • Question 1

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Mark scheme.

$\dfrac{dx}{dt} = e^t - 1$ and $\dfrac{dy}{dt} = 2e^{\frac{1}{2}t}$	**B1**
$\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = (e^t + 1)^2$ (M1 for using $\left(\dfrac{ds}{dt}\right)^2 = \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2$)	**M1A1**
Arc length is $\displaystyle\int_0^3 e^t + 1\,dt = \left[e^t + t\right]_0^3$ (good attempt at correct integral)	**M1**
$= 2 + e^3$ (or 22.1)	**A1**
	Total: 5

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Q14

9231/11, 9231/12 • May/June 2018 • Question 9

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Mark scheme.

(i) $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x$ so integral becomes $\displaystyle\int u^2\,\mathrm{d}u$	**M1**
$= \dfrac{1}{3}\tan^3 x + C$	**A1**
(ii) $I_n = \displaystyle\int_0^{\frac{1}{4}\pi} \sec^{n-2} x \sec^2 x \tan^2 x\,\mathrm{d}x$	**B1**
$= \left[\dfrac{1}{3}\sec^{n-2} x \tan^3 x\right]_0^{\frac{1}{4}\pi} - \dfrac{n-2}{3}\displaystyle\int_0^{\frac{1}{4}\pi} \sec^{n-2} x \tan^4 x\,\mathrm{d}x$	**M1**
$= \dfrac{2^{\frac{n-2}{2}}}{3} - \dfrac{n-2}{3}\displaystyle\int_0^{\frac{1}{4}\pi} \sec^{n-2} x \tan^2 x \left(\sec^2 x - 1\right)\,\mathrm{d}x$	**M1**
$= \dfrac{2^{\frac{n-2}{2}}}{3} - \dfrac{n-2}{3} I_n + \dfrac{n-2}{3} I_{n-2}$	**A1**
$\Rightarrow (3 + n - 2) I_n = \left(\sqrt{2}\right)^{n-2} + (n-2) I_{n-2}$ (AG)	**A1**
	Total: 7

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Q15

9231/11, 9231/13 • October/November 2018 • Question 4

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Mark scheme.

$\frac{dx}{dt} = 1 - \cos 2t, \frac{dy}{dt} = 2\sin t \cos t$ (B1 for $\frac{dx}{dt}$ and $\frac{dy}{dt}$.)	**B1**
$\frac{dx}{dt} = 1 - \cos 2t = 2\sin^2 t$ (Uses an appropriate identity.)	**M1**
$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4\sin^4 t + 4\sin^2 t \cos^2 t} = 2\sin t$	**A1**
$S = 2\pi \int_0^\pi (\sin^2 t)(2\sin t)\, dt \Rightarrow a = 4$ (Uses the correct formula for S)	**M1A1**
$S = 4\pi \int_0^\pi \sin^3 t\, dt = \pi \int_0^\pi 3\sin t - \sin 3t\, dt$ (Uses the result $4\sin^3 t = 3\sin t - \sin 3t$)	**M1**
$= \pi \left[-3\cos t + \frac{1}{3}\cos 3t\right]_0^\pi$	**A1**
$= \pi \left(3 - \frac{1}{3} - \left(-3 + \frac{1}{3}\right)\right) = \frac{16\pi}{3}$ (Answer must be exact.)	**A1**
	Total: 8

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Q16

9231/11, 9231/13 • October/November 2018 • Question 11

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Mark scheme.

$x=-1 \Rightarrow y^3+2y-3=0 \Rightarrow (y-1)(y^2+y+3)=0$ (Considers cubic polynomial in y.)	**M1**
There is only one real root ($=1$).	**A1**
$2x+2\left(x\dfrac{\mathrm{d}y}{\mathrm{d}x}+y\right)$ (Differentiates LHS correctly.)	**B1**
$=3y^2\dfrac{\mathrm{d}y}{\mathrm{d}x}$ (Differentiates RHS correctly.)	**B1**
$\Rightarrow (3y^2-2x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2x+2y$	
$x=-1,\ y=1 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=0.$ (Substitutes $(-1,1)$, AG.)	**B1**
$(3y^2-2x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\dfrac{\mathrm{d}y}{\mathrm{d}x}\left(6y\dfrac{\mathrm{d}y}{\mathrm{d}x}-2\right)=2+2\dfrac{\mathrm{d}y}{\mathrm{d}x}$ (Differentiates equation again.)	**M1M1**
$x=-1,\ y=1,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{2}{5}$ (Substitutes $(-1,1)$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$, AG.)	**A1**
$\displaystyle\int_{-1}^{0} x^n\frac{\mathrm{d}^ny}{\mathrm{d}x^n}\,\mathrm{d}x=\left[x^n\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\right]_{-1}^{0}-n\int_{-1}^{0}x^{n-1}\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}\,\mathrm{d}x$ (Integrates by parts.)	**M1A1**
$=(-1)^{n+1}A_{n-1}-nI_{n-1}.$ (AG.)	**A1**
$I_3=(-1)^4A_2-3I_2=A_2-3(-A_1-2I_1)=\dfrac{2}{5}+6I_1$ (Uses reduction formulae.)	**M1A1**
	Total: 13

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Q17

9231/12 • October/November 2018 • Question 11

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Mark scheme.

$I_n = \left[x(x^2-1)^n\right]_1^{\sqrt{2}} - 2n\int_1^{\sqrt{2}} x^2(x^2-1)^{n-1}\,dx$ (Integrates by parts.)	**M1A1**
$= \sqrt{2} - 2n\int_1^{\sqrt{2}} (x^2-1+1)(x^2-1)^{n-1}\,dx$ (Uses $x^2 = x^2-1+1$.)	**M1**
$= \sqrt{2} - 2nI_n - 2nI_{n-1}$	**A1**
$\Rightarrow (2n+1)I_n = \sqrt{2} - 2nI_{n-1}$ (AG.)	**A1**
$\dfrac{dx}{d\theta} = \tan\theta\sec\theta$ (Differentiates $\sec\theta$.)	**M1A1**
$\sec\theta = \sqrt{2} \Rightarrow \theta = \dfrac{\pi}{4}$ $\sec\theta = 1 \Rightarrow \theta = 0$ (Changes limits.)	**B1**
$I_n = \int_0^{\frac{\pi}{4}} (\sec^2\theta - 1)^n \tan\theta\sec\theta\,d\theta = \int_0^{\frac{\pi}{4}} \tan^{2n+1}\theta\sec\theta\,d\theta$ (Uses $\sec^2\theta - 1 = \tan^2\theta$. (AG.))	**B1**
$\int_0^{\frac{\pi}{4}} \dfrac{\sin^7\theta}{\cos^8\theta}\,d\theta = I_3$ (Deduces that integral is I_3.)	**B1**
$I_0 = \sqrt{2} - 1$ (or $I_1 = \dfrac{2-\sqrt{2}}{3}$) (Calculates I_0 or I_1)	**B1**
$3I_1 = \sqrt{2} - 2I_0 \Rightarrow I_1 = \dfrac{2-\sqrt{2}}{3}$. (Uses reduction formula to find I_1 or I_2 …)	**M1A1**
$I_2 = \dfrac{\sqrt{2}}{5} - \dfrac{4}{5}I_1 = \dfrac{7\sqrt{2}-8}{15}$ (Finds I_2.)	
$I_3 = \dfrac{\sqrt{2}}{7} - \dfrac{6}{7}I_2 = \dfrac{16-9\sqrt{2}}{35}$ (Finds I_3.)	**A1**
	Total: 14

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Q18

9231/11, 9231/12 • May/June 2019 • Question 4

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Mark scheme.

$\int_0^1 x^2 e^{x^3}\,dx = \frac{1}{3}\left[e^{x^3}\right]_0^1 = \frac{1}{3}(e-1)$	**M1A1**
$I_n = \int_0^1 x^{n-2}x^2 e^{x^3}\,dx = \left[\frac{1}{3}x^{n-2}e^{x^3}\right]_0^1 - \frac{n-2}{3}\int_0^1 x^{n-3}e^{x^3}\,dx$	**M1A1**
$\frac{e}{3} - \frac{n-2}{3}I_{n-3} \Rightarrow 3I_n = e-(n-2)I_{n-3}$	**A1**
$I_5 = \frac{1}{3}(e-3I_2) = \frac{1}{3}(e-e+1) = \frac{1}{3}$	**M1A1**
$I_8 = \frac{1}{3}(e-6I_5) = \frac{1}{3}(e-2)$.	**A1**
	Total: 8

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Q19

9231/11, 9231/12 • May/June 2019 • Question 5

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Mark scheme.

$\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{-2(\mathrm{e}^t - \mathrm{e}^{-t})}{(\mathrm{e}^t + \mathrm{e}^{-t})^2}$	**B1**
$\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{(\mathrm{e}^t + \mathrm{e}^{-t})^2 - (\mathrm{e}^t - \mathrm{e}^{-t})^2}{(\mathrm{e}^t + \mathrm{e}^{-t})^2} = \dfrac{(2\mathrm{e}^t)(2\mathrm{e}^{-t})}{(\mathrm{e}^t + \mathrm{e}^{-t})^2} = \dfrac{4}{(\mathrm{e}^t + \mathrm{e}^{-t})^2}$	**B1**
$\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \dfrac{4(\mathrm{e}^t - \mathrm{e}^{-t})^2 + 16}{(\mathrm{e}^t + \mathrm{e}^{-t})^4} = \dfrac{4(\mathrm{e}^t + \mathrm{e}^{-t})^2}{(\mathrm{e}^t + \mathrm{e}^{-t})^4}$	**M1A1**
$S = 2\pi \displaystyle\int_0^1 \left(\dfrac{\mathrm{e}^t - \mathrm{e}^{-t}}{\mathrm{e}^t + \mathrm{e}^{-t}}\right)\left(\dfrac{2}{\mathrm{e}^t + \mathrm{e}^{-t}}\right) \mathrm{d}t = 4\pi \int_0^1 \dfrac{\mathrm{e}^t - \mathrm{e}^{-t}}{(\mathrm{e}^t + \mathrm{e}^{-t})^2}\, \mathrm{d}t$	**A1**
$S = 4\pi \displaystyle\int_2^{\mathrm{e}+\mathrm{e}^{-1}} u^{-2}\, \mathrm{d}u = 4\pi \left[-u^{-1}\right]_2^{\mathrm{e}+\mathrm{e}^{-1}}$	**M1A1**
$= 4\pi \left(\dfrac{1}{2} - \dfrac{1}{\mathrm{e}+\mathrm{e}^{-1}}\right).$	**A1**
	Total: 8

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Q20

9231/13 • May/June 2019 • Question 10

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Mark scheme.

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cot^{n+1} x\right) = -(n+1)\cot^n x \csc^2 x$	**M1A1**
$= -(n+1)\cot^n x\left(\cot^2 x + 1\right) = -(n+1)\cot^{n+2} x - (n+1)\cot^n x$	**M1**
$\left[\cot^{n+1} x\right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} = -(n+1)\left(I_{n+2} + I_n\right)$	**M1**
$-1 = -(n+1)\left(I_{n+2} + I_n\right) \Rightarrow I_{n+2} = \dfrac{1}{n+1} - I_n$ (AG)	**A1**

Total: 5

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Q21

9231/11, 9231/12, 9231/13 • October/November 2019 • Question 3

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Mark scheme.

$I_{n+2} = \left[\dfrac{x^{-n-1}}{-n-1}\sin\pi x\right]_{\frac{1}{2}}^{1} - \pi\displaystyle\int_{\frac{1}{2}}^{1}\dfrac{x^{-n-1}}{-n-1}\cos\pi x\,\mathrm{d}x$	**M1A1**
Integrates by parts.	
$= \dfrac{2^{n+1}}{n+1} + \dfrac{\pi}{n+1}\left(\left[\dfrac{x^{-n}}{-n}\cos\pi x\right]_{\frac{1}{2}}^{1} + \pi\displaystyle\int_{\frac{1}{2}}^{1}\dfrac{x^{-n}}{-n}\sin\pi x\,\mathrm{d}x\right)$	**M1**
Integrates by parts again.	
$= \dfrac{2^{n+1}}{n+1} + \dfrac{\pi}{n+1}\left(\dfrac{1}{n} - \dfrac{\pi}{n}I_n\right) \Rightarrow (n+1)I_{n+2} = 2^{n+1} + \pi\left(\dfrac{1}{n} - \dfrac{\pi}{n}I_n\right)$	**M1**
Uses I_n.	
$\Rightarrow n(n+1)I_{n+2} = 2^{n+1}n + \pi - \pi^2 I_n$	**A1**
AG	
$2I_3 = 4 + \pi - \pi^2 I_1$	**M1**
$12I_5 = 48 + \pi - \dfrac{\pi^2}{2}(4 + \pi - \pi^2 I_1)$	
Substitutes I_3 into reduction formula.	
$\Rightarrow I_5 = 4 + \dfrac{1}{24}(2\pi - 4\pi^2 - \pi^3 + \pi^4 I_1)$	**A1**
AEF, must be exact with fractions simplified.	
	Total: 7

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Q22

9231/21, 9231/22 • May/June 2020 • Question 4

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Mark scheme.

(a)

$\int_0^1 x^2\,dx < \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$	**M1 A1**
$\frac{1}{n^3}\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6n^3} = \frac{2n^2+3n+1}{6n^2}$	**M1 A1**

(b)

$\int_0^1 x^2\,dx > \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^2$	**M1 A1**
$= \frac{1}{n^3}\sum_{r=1}^{n-1} r^2 = \frac{(n-1)n(2n-2+1)}{6n^3} = \frac{2n^2-3n+1}{6n^2}$	**M1 A1**

Total: 8

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Q23

9231/21, 9231/22 • May/June 2020 • Question 6

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Mark scheme.

(a)	
$I_1 = \int_0^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}}\,\mathrm{d}x = \left[\sin^{-1} x\right]_0^{\frac{1}{2}} = \frac{1}{6}\pi$	**M1 A1**
(b)	
$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x^2)^{-\frac{1}{2}n}\right) = nx^2(1-x^2)^{-\frac{1}{2}n-1} + (1-x^2)^{-\frac{1}{2}n}$	**M1 A1**
$= n\left(1-(1-x^2)\right)(1-x^2)^{-\frac{1}{2}n-1} + (1-x^2)^{-\frac{1}{2}n}$	**M1**
$\left[x(1-x^2)^{-\frac{1}{2}n}\right]_0^{\frac{1}{2}} = nI_{n+2} - nI_n + I_n$	**M1**
$\frac{1}{2}\left(\frac{3}{4}\right)^{-\frac{1}{2}n} = nI_{n+2} - (n-1)I_n \Rightarrow nI_{n+2} = 2^{n-1}3^{-\frac{1}{2}n} + (n-1)I_n$	**A1**
(c)	
$I_3 = 3^{-\frac{1}{2}}$	**B1**
$3I_5 = 2^2 3^{-\frac{3}{2}} + 2I_3 \Rightarrow I_5 = \frac{10}{27}\sqrt{3}$	**M1 A1**
	Total: 10

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Q24

9231/23 • May/June 2020 • Question 2

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Mark scheme.

(a)	
$I_n = \left[-\frac{1}{3}e^{-3x}(1+3x)^n\right]_0^1 + n\int_0^1 e^{-3x}(1+3x)^{n-1}\,dx$	**M1 A1**
$-\frac{1}{3}e^{-3}4^n + \frac{1}{3} + nI_{n-1} \Rightarrow 3I_n = 1 - 4^n e^{-3} + 3nI_{n-1}$	**A1**
(b)	
$I_0 = \int_0^1 e^{-3x}\,dx = -\frac{1}{3}\left[e^{-3x}\right]_0^1 = \frac{1}{3}(1-e^{-3})$	**B1**
$I_1 = \frac{1}{3}(1-4e^{-3}+1-e^{-3}) = \frac{1}{3}(2-5e^{-3})$; $I_2 = \frac{1}{3}(1-16e^{-3}+2(2-5e^{-3}))$	**M1**
$I_2 = \frac{1}{3}(5-26e^{-3})$	**A1**
	Total: 6

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Q25

9231/23 • May/June 2020 • Question 4

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Mark scheme.

(a)	
$\ln 2 + \ln 3 + \cdots + \ln N$	**M1**
$> \displaystyle\int_1^N \ln x \, \mathrm{d}x$	**M1**
$\displaystyle\int_1^N \ln x \, \mathrm{d}x = [x \ln x]_1^N - \int_1^N 1 \, \mathrm{d}x = N \ln N - N + 1$	**M1 A1**
$\ln N! > N \ln N - N + 1$	**A1**
(b)	
$\ln 1 + \ln 2 + \cdots + \ln(N-1) < \displaystyle\int_1^N \ln x \, \mathrm{d}x$	**M1**
(or $\ln 2 + \cdots + \ln(N-1) < \int_2^N \ln x \, \mathrm{d}x$)	**M1**
$\ln N! < N \ln N - N + 1 + \ln N = (N+1) \ln N - N + 1$	**A1**
	Total: 8

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Q26

9231/23 • May/June 2020 • Question 5

↑ Back to contents

Mark scheme.

(a)	
$\dot{x} = t - t^{-1}$, $\dot{y} = 2$	**B1**
$\dot{x}^2 + \dot{y}^2 = t^2 - 2 + t^{-2} + 4 = \left(t + t^{-1}\right)^2$	**M1 A1**
$\int_{\frac{1}{2}}^{2} t + t^{-1}\,\mathrm{d}t = \left[\frac{1}{2}t^2 + \ln t\right]_{\frac{1}{2}}^{2} = \frac{15}{8} + 2\ln 2$	**M1 A1**
(b)	
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}} = \frac{2}{t - t^{-1}}$	**B1**
$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{t - t^{-1}}\right) = \frac{-2(1 + t^{-2})}{(t - t^{-1})^2}$	**B1**
$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2}{t - t^{-1}}\right) \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-2(1 + t^{-2})}{(t - t^{-1})^3}$	**M1 A1**
	Total: 9

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Q27

9231/21, 9231/23 • October/November 2020 • Question 4

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Mark scheme.

Forms the sum of the areas of the rectangles: $\int_0^1 1-x^3\,\mathrm{d}x \leqslant \frac{1}{n}+\frac{1}{n}\left(1-\frac{1}{n^3}\right)+\cdots+\frac{1}{n}\left(1-\frac{(n-1)^3}{n^3}\right)$.	**M1 A1**
Applies $\sum_{r=1}^{n-1} r^3 = \frac{1}{4}(n-1)^2n^2$: $= 1-\frac{1}{n^4}\sum_{r=1}^{n-1} r^3 = 1-\frac{(n-1)^2n^2}{4n^4}$.	**M1**
AG: $= \frac{4n^2-(n-1)^2}{4n^2} = \frac{3n^2+2n-1}{4n^2}$.	**A1**
(b) Forms the sum of the areas of appropriate rectangles: $\int_0^1 1-x^3\,\mathrm{d}x \geqslant \frac{1}{n}\left(1-\frac{1}{n^3}\right)+\frac{1}{n}\left(1-\frac{2^3}{n^3}\right)+\cdots+\frac{1}{n}\left(1-\frac{(n-1)^3}{n^3}\right)$.	**M1 A1**
Applies $\sum_{r=1}^{n-1} r^3 = \frac{1}{4}(n-1)^2n^2$: $= \frac{n-1}{n}-\frac{1}{n^4}\sum_{r=1}^{n-1} r^3 = \frac{n-1}{n}-\frac{(n-1)^2n^2}{4n^4}$.	**M1**
$= \frac{4n(n-1)-(n-1)^2}{4n^2} = \frac{3n^2-2n-1}{4n^2}$.	**A1**
	Total: 8

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Q28

9231/22 • October/November 2020 • Question 8

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Mark scheme.

Forms sum of the areas of the rectangles, need to see consideration of areas, not just the sum given in the question: Sum $= \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{7}} + \cdots + \dfrac{1}{\sqrt{n^2+n+1}}$.	**M1 A1**
Compares with integral. Limits have to be correct: $< \displaystyle\int_0^n \frac{1}{\sqrt{x^2+x+1}}\,\mathrm{d}x$.	**M1**
Completes the square: $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$.	**B1**
Uses formula: $\displaystyle\int_0^n \frac{1}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}\,\mathrm{d}x = \left[\sinh^{-1}\left(\frac{x+\frac{1}{2}}{\frac{1}{2}\sqrt{3}}\right)\right]_0^n = \left[\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]_0^n$.	**M1 A1**
Uses logarithmic form of $\sinh^{-1}$: $= \left[\ln\left(\dfrac{2x+1}{\sqrt{3}} + \sqrt{\dfrac{(2x+1)^2+3}{3}}\right)\right]_0^n = \left[\ln\left(\dfrac{2x+1+2\sqrt{x^2+x+1}}{\sqrt{3}}\right)\right]_0^n$.	**M1 A1**
Inserts limits: $= \ln\left(\dfrac{2n+1+2\sqrt{n^2+n+1}}{\sqrt{3}}\right) - \ln\left(\dfrac{3}{\sqrt{3}}\right)$.	**M1**
AG: $= \ln\left(\dfrac{1}{3} + \dfrac{2}{3}n + \dfrac{2}{3}\sqrt{n^2+n+1}\right)$.	**A1**
	Total: 10

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Q29

9231/21, 9231/22 • May/June 2021 • Question 3

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Mark scheme.

(a)	
Forms the sum of the areas of the n rectangles: $\int_0^1 x^3\,\mathrm{d}x < \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^3$ (need last term for A1).	**M1 A1**
Applies $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$: $\frac{1}{n^4}\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4n^4} = \left(\frac{n+1}{2n}\right)^2$ (AG).	**M1 A1**
(b)	
Forms the sum of the areas of appropriate rectangles (need the last term for A1).	**M1 A1**
Applies $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$: $\frac{1}{n^4}\sum_{r=1}^{n-1} r^3 = \frac{(n-1)^2n^2}{4n^4} = \left(\frac{n-1}{2n}\right)^2$ (accept $\left(\frac{n+1}{2n}\right)^2 - \frac{1}{n}$).	**M1 A1**
(c)	
Simplifies their $U_n - L_n$ to $\frac{k}{n}$: $\left(\frac{n+1}{2n}\right)^2 - \left(\frac{n-1}{2n}\right)^2 = \frac{1}{n} < 10^{-3}$ leading to $n > 10^3$.	**M1**
Least value of n is 1001.	**A1**
	Total: 10

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Q30

9231/23 • May/June 2021 • Question 3

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Mark scheme.

(a)	
Compares with sum of the areas of the rectangles: $\sum_{r=1}^{N} \frac{r}{2r^2-1} = 1 + \sum_{r=2}^{N} \frac{r}{2r^2-1}$.	**M1 A1**
Compares with integral: $< 1 + \int_1^N \frac{x}{2x^2-1}\,dx$.	**M1**
Finds integral: $\int_1^N \frac{x}{2x^2-1}\,dx = \left[\frac{1}{4}\ln(2x^2-1)\right]_1^N$.	**M1 A1**
Inserts limits (AG): $\sum_{r=1}^{N} \frac{r}{2r^2-1} < 1 + \frac{1}{4}\ln(2N^2-1)$.	**M1 A1**
(b)	
Compares with integral: $\sum_{r=1}^{N} \frac{r}{2r^2-1} = \sum_{r=1}^{N-1} \frac{r}{2r^2-1} + \frac{N}{2N^2-1} > \int_1^N \frac{x}{2x^2-1}\,dx + \frac{N}{2N^2-1}$.	**M1 A1**
$\frac{1}{4}\ln(2N^2-1) + \frac{N}{2N^2-1}$ (CWO).	**A1**
	Total: 10

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Q31

9231/23 • May/June 2021 • Question 7

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Mark scheme.

(a)	
Recognises integral or uses appropriate substitution: $I_1 = \int_0^{3/2} (4+x^2)^{-1/2} = \left[\sinh^{-1}\left(\frac{1}{2}x\right)\right]_0^{3/2}$.	**M1 A1**
$\ln 2$ (insert limits, must simplify).	**A1**
(b)	
Uses the product rule to differentiate: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(4+x^2)^{-\frac{1}{2}n}\right) = -nx^2(4+x^2)^{-\frac{1}{2}n-1} + (4+x^2)^{-\frac{1}{2}n}$.	**M1 A1**
Applies $x^2 = 4 + x^2 - 4$: $-n(4+x^2-4)(4+x^2)^{-\frac{1}{2}n-1} + (4+x^2)^{-\frac{1}{2}n}$.	**M1**
Integrates both sides using the limits given: $\left[x(4+x^2)^{-\frac{1}{2}n}\right]_0^{3/2} = -nI_n + 4nI_{n+2} + I_n$.	**M1**
Substitutes limits and rearranges (AG): $\frac{3}{2}\left(\frac{2}{5}\right)^n = (1-n)I_n + 4nI_{n+2} \Rightarrow 4nI_{n+2} = \frac{3}{2}\left(\frac{2}{5}\right)^n + (n-1)I_n$.	**A1**
(c)	
Applies reduction formula with $n = 1$: $I_3 = \frac{3}{20}$.	**B1**
Applies reduction formula with $n = 3$: $12I_5 = \frac{3}{2}\left(\frac{8}{125}\right) + 2I_3 \Rightarrow I_5 = \frac{33}{1000} = 0.033$.	**M1 A1**
	Total: 11

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Q32

9231/21, 9231/23 • October/November 2021 • Question 4

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Mark scheme.

$\sum_{r=1}^{N}\frac{\ln r}{r^2}=\frac{\ln 2}{4}+\sum_{r=3}^{N}\frac{\ln r}{r^2}$. Compares with sum of areas of rectangles.	**M1 A1**
$<\frac{\ln 2}{4}+\int_2^N \frac{\ln x}{x^2}\,\mathrm{d}x$. Compares with integral.	**M1**
$\int_2^N \frac{\ln x}{x^2}\,\mathrm{d}x=\left[-\frac{\ln x+1}{x}\right]_2^N$. Finds integral.	**M1 A1**
$\sum_{r=1}^{N}\frac{\ln r}{r^2}<\frac{\ln 2}{4}+\left(-\frac{\ln N+1}{N}+\frac{\ln 2+1}{2}\right)=\frac{2+3\ln 2}{4}-\frac{1+\ln N}{N}$ (AG).	**M1 A1**
Part (b): compares with integral (lower limit of 1 scores M0): $\sum_{r=1}^{N}\frac{\ln r}{r^2}>\int_2^N \frac{\ln x}{x^2}\,\mathrm{d}x+\frac{\ln N}{N^2}$.	**M1 A1**
$=\frac{\ln 2+1}{2}-\frac{\ln N+1}{N}+\frac{\ln N}{N^2}$.	**A1**
	Total: 10

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Q33

9231/22 • October/November 2021 • Question 3

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Mark scheme.

Forms the sum of the areas of the rectangles (three terms, including last, for A1; allow use of sigma): $\int_0^1 1-x^2\,dx < \frac{1}{n}+\frac{1}{n}\left(1-\frac{1}{n^2}\right)+\cdots+\frac{1}{n}\left(1-\frac{(n-1)^2}{n^2}\right)$.	**M1 A1**
Applies $\sum_{r=1}^{n-1} r^2 = \frac{1}{6}(n-1)n(2n-1)$: $1-\frac{1}{n^3}\sum_{r=1}^{n-1} r^2 = 1-\frac{(n-1)n(2n-1)}{6n^3}$.	**M1**
$\frac{6n^2-(n-1)(2n-1)}{6n^2} = \frac{4n^2+3n-1}{6n^2}$ (AG).	**A1**
Part (b): forms sum of areas of appropriate rectangles (three terms, including last).	**M1 A1**
Applies $\sum_{r=1}^{n-1} r^2 = \frac{1}{6}(n-1)n(2n-1)$.	**M1**
$\frac{6n(n-1)-(n-1)(2n-1)}{6n^2} = \frac{4n^2-3n-1}{6n^2}$.	**A1**
	Total: 8

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Q34

9231/22 • October/November 2021 • Question 5(a)

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Mark scheme.

$\dot{x} = 3 - 2t^{-2} + 2t^2$, $\dot{y} = 4 + \frac{3}{2}t^{-2} - \frac{3}{2}t^2$. Differentiates x and y with respect to t.	**B1**
Expands $\dot{x}^2 + \dot{y}^2$: $4t^4 + 12t^2 + \dfrac{4}{t^4} - \dfrac{12}{t^2} + 1 + \dfrac{23}{2} + \dfrac{9t^4}{4} + \dfrac{12}{t^2} + \dfrac{9}{4t^4} - 12t^2$.	**M1**
Factorises: $\dfrac{25}{4}t^4 + \dfrac{25}{4}t^{-4} + \dfrac{25}{2} = \dfrac{25}{4}(t^2 + t^{-2})^2$ (AG).	**M1 A1**
Applies correct arc-length formula: $\dfrac{5}{2}\displaystyle\int_1^2 t^2 + t^{-2}\,\mathrm{d}t = \dfrac{5}{2}\left[\frac{1}{3}t^3 - t^{-1}\right]_1^2 = \dfrac{85}{12}$.	**M1 A1**
	Total: 6

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Q35

9231/22 • October/November 2021 • Question 8(c)(d)

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Mark scheme.

$I_n = \int_0^{\ln 3} \operatorname{sech}^{n-2} x \operatorname{sech}^2 x \tanh^2 x \, dx$. Separates into correct structure.	**B1**
$\left[\frac{1}{3} \operatorname{sech}^{n-2} x \tanh^3 x\right]_0^{\ln 3} + \frac{1}{3}(n-2) \int_0^{\ln 3} \operatorname{sech}^{n-2} x \tanh^4 x \, dx$. Uses integration by parts correctly.	**M1**
$\frac{1}{3}\left(\frac{3}{5}\right)^{n-2}\left(\frac{4}{5}\right)^3 + \frac{1}{3}(n-2)(I_{n-2} - I_n)$. Uses $1 - \operatorname{sech}^2 x = \tanh^2 x$.	**M1 A1**
$(3+n-2)I_n = \left(\frac{3}{5}\right)^{n-2}\left(\frac{4}{5}\right)^3 + (n-2)I_{n-2}$ leading to $(n+1)I_n = \left(\frac{3}{5}\right)^{n-2}\left(\frac{4}{5}\right)^3 + (n-2)I_{n-2}$ (AG).	**A1**
$I_2 = \frac{1}{3}\left(\frac{4}{5}\right)^3 = \frac{64}{375}$.	**B1**
$(4+1)I_4 = \left(\frac{3}{5}\right)^2\left(\frac{4}{5}\right)^3 + 2I_2$ leading to $I_4 = \dfrac{4928}{46875} = 0.105$.	**M1 A1**
	Total: 8

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Q36

9231/21, 9231/22 • May/June 2022 • Question 1

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Mark scheme.

Forms s correctly: $s = \int_0^\alpha \sqrt{e^{\frac{3}{2}\theta} + \frac{9}{16}e^{\frac{3}{2}\theta}}\, d\theta = \frac{5}{4}\int_0^\alpha e^{\frac{3}{4}\theta}\, d\theta$	**M1 A1**
Integrates: $s = \frac{5}{3}\left[e^{\frac{3}{4}\theta}\right]_0^\alpha = \frac{5}{3}\left(e^{\frac{3}{4}\alpha} - 1\right)$	**M1 A1**
$\alpha = \frac{4}{3}\ln\left(1 + \frac{3}{5}s\right)$	**A1**
	Total: 5

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Q37

9231/21, 9231/22 • May/June 2022 • Question 4

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Mark scheme.

Forms the sum of the areas of the rectangles: $\int_0^1 2^x\,dx < \left(\frac{1}{N}\right)2^{\frac{1}{N}} + \left(\frac{1}{N}\right)2^{\frac{2}{N}} + \cdots + \left(\frac{1}{N}\right)2^{\frac{N-1}{N}} + \left(\frac{1}{N}\right)2^{\frac{N}{N}}$	**(a) M1 A1**
Applies $\sum_{n=1}^{N} r^n = \frac{r(r^N-1)}{r-1}$, AG: $= \frac{1}{N}\sum_{n=1}^{N}\left(2^{\frac{1}{N}}\right)^n = \frac{2^{\frac{1}{N}}}{N\left(2^{\frac{1}{N}}-1\right)}$	**M1 A1**
Forms the sum of the areas of appropriate rectangles: $\int_0^1 2^x\,dx > \left(\frac{1}{N}\right) + \left(\frac{1}{N}\right)2^{\frac{1}{N}} + \cdots + \left(\frac{1}{N}\right)2^{\frac{N-2}{N}} + \left(\frac{1}{N}\right)2^{\frac{N-1}{N}}$	**(b) M1 A1**
Applies $\sum_{n=0}^{N-1} r^n = \frac{r^N-1}{r-1}$: $= \frac{1}{N}\sum_{n=0}^{N-1}\left(2^{\frac{1}{N}}\right)^n = \frac{1}{N\left(2^{\frac{1}{N}}-1\right)}$	**M1 A1**
Simplifies $U_N - L_N$ to $\frac{c}{n}$: $\frac{2^{\frac{1}{N}}}{N\left(2^{\frac{1}{N}}-1\right)} - \frac{1}{N\left(2^{\frac{1}{N}}-1\right)} = \frac{1}{N} < 10^{-4}$ leading to $N > 10^4$	**(c) M1**
Least value of N is 10001	**A1**
	Total: 10

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Q38

9231/23 • May/June 2022 • Question 2(b)

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Mark scheme.

Forms correct integral: $\int_0^{\frac{1}{4}\pi} \sqrt{1+\tan^2 x}\, dx$	**M1**
$= \int_0^{\frac{1}{4}\pi} \sec x\, dx$	**A1**
Integrates (answer given in List MF19; accept $\ln(\sec x + \tan x)$): $= \left[\ln\left(\tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right)\right)\right]_0^{\frac{1}{4}\pi}$	**M1**
Accept $\ln(\sqrt{2}+1)$: $\ln\tan\left(\frac{3}{8}\pi\right) = 0.881$	**A1**
	Total: 4

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Q39

9231/23 • May/June 2022 • Question 6

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Mark scheme.

Forms the sum of the areas of the rectangles: $\int_0^1 \ln(1+x)\,dx < \left(\frac{1}{n}\right)\ln\left(1+\frac{1}{n}\right)+\left(\frac{1}{n}\right)\ln\left(1+\frac{2}{n}\right)+\cdots+\left(\frac{1}{n}\right)\ln\left(1+\frac{n}{n}\right)$	**(a) M1 A1**
Applies laws of logarithms, AG: $=\frac{1}{n}\big(n\ln\frac{1}{n}+\ln(n+1)+\ln(n+2)+\cdots+\ln(2n)\big)=\frac{1}{n}\ln\frac{(2n)!}{n!}-\ln n$	**M1 A1**
Forms the sum of the areas of appropriate rectangles: $\int_0^1 \ln(1+x)\,dx > \left(\frac{1}{n}\right)\ln\left(1+\frac{1}{n}\right)+\left(\frac{1}{n}\right)\ln\left(1+\frac{2}{n}\right)+\cdots+\left(\frac{1}{n}\right)\ln\left(1+\frac{n-1}{n}\right)$	**(b) M1 A1**
Applies laws of logarithms: $=\frac{1}{n}\big((n-1)\ln\frac{1}{n}+\ln(n+1)+\ln(n+2)+\cdots+\ln(2n-1)\big)$	**M1**
Equivalent answers accepted: $\frac{1}{n}\ln\frac{(2n-1)!}{n!}-\left(\frac{n-1}{n}\right)\ln n$	**A1**
Simplifies their U_n-L_n to $\frac{c}{n}$: $U_n-L_n=\frac{\ln 2}{n}$	**(c) M1**
AG: $\frac{\ln 2}{n}\to 0$ as $n\to\infty$	**A1**
	Total: 10

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Q40

9231/23 • May/June 2022 • Question 8

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Mark scheme.

Allow with $+C$ missing: $\displaystyle\int \sin\theta\cos^n\theta\,d\theta = -\frac{\cos^{n+1}\theta}{n+1}\,(+C)$	**(a) M1 A1**
Integrates by parts using part (a) (allow with limits missing): $\displaystyle\int_0^{\frac{1}{2}\pi}\sin^{m-1}\theta\sin\theta\cos^n\theta\,d\theta = \left[-\frac{\sin^{m-1}\theta\cos^{n+1}\theta}{n+1}\right]_0^{\frac{1}{2}\pi} + \frac{m-1}{n+1}\int_0^{\frac{1}{2}\pi}\cos^{n+2}\theta\sin^{m-2}\theta\,d\theta$	**(b) M1 A1**
Applies $\cos^2\theta = 1-\sin^2\theta$: $= \dfrac{m-1}{n+1}\int_0^{\frac{1}{2}\pi}\cos^n\theta\sin^{m-2}\theta(1-\sin^2\theta)\,d\theta$	**M1**
Combines with LHS: $I_{m,n} = \dfrac{m-1}{n+1}I_{m-2,n} - \dfrac{m-1}{n+1}I_{m,n}$	**M1**
AG: $\dfrac{n+m}{n+1}I_{m,n} = \dfrac{m-1}{n+1}I_{m-2,n}$ leading to $I_{m,n} = \dfrac{m-1}{m+n}I_{m-2,n}$	**A1**
$z + z^{-1} = 2\cos\theta$	**(c) B1**
Expands and groups (A0 if no grouping shown): $(z+z^{-1})^5 = (z^5+z^{-5}) + 5(z^3+z^{-3}) + 10(z+z^{-1})$	**M1 A1**
Substitutes $z^n + z^{-n} = 2\cos n\theta$: $2^5\cos^5\theta = 2\cos5\theta + 5(2\cos3\theta) + 10(2\cos\theta)$	**M1**
$\cos^5\theta = \frac{1}{16}\cos5\theta + \frac{5}{16}\cos3\theta + \frac{5}{8}\cos\theta$	**A1**
Applies reduction formula in part (b): $I_{2,5} = \frac{1}{7}I_{0,5}$	**(d) M1**
Applies part (c): $\frac{1}{7}I_{0,5} = \frac{1}{7}\int_0^{\frac{1}{2}\pi}\cos^5\theta\,d\theta = \frac{1}{7}\int_0^{\frac{1}{2}\pi}\frac{1}{16}\cos5\theta + \frac{5}{16}\cos3\theta + \frac{5}{8}\cos\theta\,d\theta$	**M1**
$= \dfrac{1}{7}\left[\frac{1}{80}\sin5\theta + \frac{5}{48}\sin3\theta + \frac{5}{8}\sin\theta\right]_0^{\frac{1}{2}\pi}$	**A1**
$= \dfrac{8}{105}$	**A1**
	Total: 16

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Q41

9231/21, 9231/23 • October/November 2022 • Question 3

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Mark scheme.

$\dot{x} = \mathrm{e}^t - t^2$.	**B1**
$\dot{y} = 2t\mathrm{e}^{\frac{1}{2}t}$.	**B1**
$\sqrt{(\mathrm{e}^t - t^2)^2 + 4t^2\mathrm{e}^t} = \sqrt{\mathrm{e}^{2t} + 2t^2\mathrm{e}^t + t^4} = \mathrm{e}^t + t^2$. Finds $\sqrt{\dot{x}^2 + \dot{y}^2}$.	**M1 A1**
$\int_0^2 \mathrm{e}^t + t^2 \, \mathrm{d}t = \left[\mathrm{e}^t + \frac{1}{3}t^3\right]_0^2 = \mathrm{e}^2 + \frac{5}{3}$.	**M1 A1**
	Total: 6

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Q42

9231/22 • October/November 2022 • Question 3(a)

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Mark scheme.

$\sqrt{1+(\mathrm{e}^x-\frac{1}{4}\mathrm{e}^{-x})^2} = \sqrt{1+\mathrm{e}^{2x}-\frac{1}{2}+\frac{1}{16}\mathrm{e}^{-2x}} = \sqrt{\mathrm{e}^{2x}+\frac{1}{2}+\frac{1}{16}\mathrm{e}^{-2x}}$. Finds $\sqrt{1+(y')^2}$.	**M1**
$= \mathrm{e}^x + \frac{1}{4}\mathrm{e}^{-x}$.	**A1**
$2\pi\int_0^1 (\mathrm{e}^x+\frac{1}{4}\mathrm{e}^{-x})(\mathrm{e}^x+\frac{1}{4}\mathrm{e}^{-x})\,\mathrm{d}x$. Correct formula for surface area with limits.	**M1**
$2\pi\int_0^1 \mathrm{e}^{2x}+\frac{1}{2}+\frac{1}{16}\mathrm{e}^{-2x}\,\mathrm{d}x$. Expanded integrand.	**A1**
$2\pi\left[\frac{1}{2}\mathrm{e}^{2x}+\frac{1}{2}x-\frac{1}{32}\mathrm{e}^{-2x}\right]_0^1$. Integrates and substitutes limits.	**M1**
$2\pi\left(\frac{1}{2}\mathrm{e}^2-\frac{1}{32}\mathrm{e}^{-2}+\frac{1}{32}\right)=\pi\left(\mathrm{e}^2-\frac{1}{16}\mathrm{e}^{-2}+\frac{1}{16}\right)$.	**A1**
	Total: 6

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Q43

9231/22 • October/November 2022 • Question 6

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Mark scheme.

Sum $= \dfrac{1}{\sqrt{8}} + \dfrac{1}{\sqrt{15}} + \cdots + \dfrac{1}{\sqrt{n^2+2n}} = \displaystyle\sum_{r=2}^{n} \dfrac{1}{\sqrt{r^2+2r}}$. Forms sum of areas of rectangles (at least three terms including first and last for A1).	**M1 A1**
$< \displaystyle\int_1^n \dfrac{1}{\sqrt{x^2+2x}}\,\mathrm{d}x$. Compares with integral.	**M1**
$x^2+2x = (x+1)^2-1$. Completes the square.	**B1**
$\displaystyle\int_1^n \dfrac{1}{\sqrt{(x+1)^2-1}}\,\mathrm{d}x = \left[\cosh^{-1}(x+1)\right]_1^n$. Uses formula; correct limits for A1.	**M1 A1**
$= \left[\ln\left(x+1+\sqrt{x^2+2x}\right)\right]_1^n$. Uses logarithmic form of $\cosh^{-1}$.	**M1 A1**
$= \ln(n+1+\sqrt{n^2+2n}) - \ln(2+\sqrt{3})$. Inserts limits.	**M1**
$\displaystyle\sum_{r=1}^{n} \dfrac{1}{\sqrt{r^2+2r}} < \ln(n+1+\sqrt{n^2+2n}) - \ln(2+\sqrt{3}) + \tfrac{1}{3}\sqrt{3}$. Adds term from $r=1$. AG.	**A1**
	Total: 10

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Q44

9231/21, 9231/22 • May/June 2023 • Question 4

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Mark scheme.

Uses the product rule to differentiate: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x(1+x^5)^n\right) = 5nx^5(1+x^5)^{n-1} + (1+x^5)^n$.	**M1 A1**
Uses $x^5 = x^5 + 1 - 1$: $5n(1+x^5-1)(1+x^5)^{n-1} + (1+x^5)^n$.	**M1***
Integrates both sides using the limits given (requires previous method mark): $\left[x(1+x^5)^n\right]_0^1 = 5nI_n - 5nI_{n-1} + I_n$.	**DM1**
Substitutes limits and rearranges (AG): $2^n = (5n+1)I_n - 5nI_{n-1} \Rightarrow (5n+1)I_n = 2^n + 5nI_{n-1}$. (Alternative: integrate by parts directly, using $x^5 = x^5 + 1 - 1$ to form recursive formula, then substitute limits and rearrange — same mark allocation.) **(b)**	**A1**
$I_1 = \left[x + \frac{1}{6}x^6\right]_0^1 = \frac{7}{6}$ (or $I_0 = 1$).	**B1**
Applies reduction formula: $11I_2 = 2^2 + 10I_1 \Rightarrow I_2 = \frac{47}{33}$.	**M1 A1**
$16I_3 = 2^3 + 15I_2 \Rightarrow I_3 = \frac{323}{176}$.	**A1**
	Total: 9

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Q45

9231/23 • May/June 2023 • Question 6

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Mark scheme.

Forms the sum of the areas of the rectangles (must have first two terms and last term for A1; accept $\left(\frac{1}{n}\right)\left(1-\frac{n}{n}\right)^2$ for the last term; accept summation form with correct limits): $\int_0^1 (1-x)^2\,\mathrm{d}x < \left(\frac{1}{n}\right)\left(1-\frac{0}{n}\right)^2 + \left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(1-\frac{n-1}{n}\right)^2$.	**M1 A1**
Setting up correct series so that formulae from MF19 can be applied (can also recognise sum as $\frac{1}{n^3}\sum_{r=1}^{n} r^2$): $\frac{1}{n} + \frac{1}{n^3}\sum_{r=1}^{n-1}(n-r)^2 = \frac{1}{n} + \frac{1}{n^3}\sum_{r=1}^{n-1}(n^2 - 2nr + r^2)$.	**M1 A1**
AG (must have gained previous accuracy mark): $\frac{1}{n} + \frac{n^2(n-1)}{n^3} - \frac{n^2(n-1)}{n^3} + \frac{n(n-1)(2n-1)}{6n^3} = \frac{1}{n} + \frac{2n^2-3n+1}{6n^2} = \frac{2n^2+3n+1}{6n^2}$. **(b)**	**A1**
Forms the sum of the areas of appropriate rectangles (must have first two and last terms for A1; accept summation form with correct limits): $\int_0^1 (1-x)^2\,\mathrm{d}x > \left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)\left(1-\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(1-\frac{n-1}{n}\right)^2$.	**M1 A1**
Setting up correct series so that formulae from MF19 can be applied (recognising sum as $\frac{1}{n^3}\sum_{r=1}^{n-1} r^2 = \frac{1}{n^3}\sum_{r=1}^{n} r^2 - \frac{1}{n}$ without wrong working scores M1 A1 M1): $\frac{1}{n^3}\sum_{r=1}^{n-1}(n^2 - 2nr + r^2) = \frac{n^2(n-1)}{n^3} - \frac{n^2(n-1)}{n^3} + \frac{n(n-1)(2n-1)}{6n^3}$.	**M1**
$L_n = \dfrac{2n^2-3n+1}{6n^2}$. **(c)**	**A1**
Simplifies $U_n - L_n$ to $\frac{c}{n}$: $U_n - L_n = \frac{1}{n}$.	**M1**
AG: $\frac{1}{n} \to 0$ as $n \to \infty$.	**A1**
	Total: 11

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Q46

9231/23 • May/June 2023 • Question 7

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Mark scheme.

Recognises integral: $I_{-1} = \int_0^{4/3} (1+x^2)^{-\frac{1}{2}}\,\mathrm{d}x = \left[\sinh^{-1} x\right]_0^{4/3} = \ln 3$. **(b)**	**M1 A1**
Uses the product rule to differentiate: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1+x^2)^{\frac{1}{2}n}\right) = nx^2(1+x^2)^{\frac{1}{2}n-1} + (1+x^2)^{\frac{1}{2}n}$.	**M1 A1**
Uses $x^2 = 1 + x^2 - 1$: $n(1+x^2-1)(1+x^2)^{\frac{1}{2}n-1} + (1+x^2)^{\frac{1}{2}n}$.	**M1***
Integrates both sides using the limits given (requires previous method mark): $\left[x(1+x^2)^{\frac{1}{2}n}\right]_0^{4/3} = nI_n - nI_{n-2} + I_n$.	**DM1**
Substitutes limits and rearranges (AG): $\frac{4}{3}\left(\frac{5}{3}\right)^n = (n+1)I_n - nI_{n-2} \Rightarrow (n+1)I_n = nI_{n-2} + \frac{4}{3}\left(\frac{5}{3}\right)^n$. **(c)**	**A1**
Forms correct integral with correct limits: $s = \int_0^{2/3} \sqrt{1+4x^2}\,\mathrm{d}x$.	**M1**
AG: $\frac{1}{2}\int_0^{4/3} \sqrt{1+u^2}\,\mathrm{d}u = \frac{1}{2}I_1$.	**A1**
Applies reduction formula with $n=1$: $2I_1 = I_{-1} + \frac{4}{3}\left(\frac{5}{3}\right)$.	**M1**
$s = \frac{1}{4}\ln 3 + \frac{5}{9}$.	**A1**
	Total: 11

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Q47

9231/21, 9231/23 • October/November 2023 • Question 5(a)

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Mark scheme.

Differentiates x and y with respect to t: $\dot{x} = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$, $\dot{y} = 2$.	**B1**
Factorises $\dot{x}^2 + \dot{y}^2$: $\dot{x}^2 + \dot{y}^2 = \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)^2 + 4 = t + 2 + t^{-1} = \left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right)^2$.	**M1 A1**
Applies correct formula for arc length. M1 for their $\int_0^3 \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}t$. Answer must be simplified to $4\sqrt{3}$ for A1: $\int_0^3 t^{\frac{1}{2}} + t^{-\frac{1}{2}}\,\mathrm{d}t = \left[\frac{2}{3}t^{\frac{3}{2}} + 2t^{\frac{1}{2}}\right]_0^3 = 4\sqrt{3}$.	**M1 A1**
	Total: 5

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Q48

9231/21, 9231/23 • October/November 2023 • Question 8

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Mark scheme.

(a)	
$\dfrac{z^n-1}{z-1}$.	**B1**
(b)	
$\dfrac{z^n-1}{z-1}=\dfrac{\cos n\theta-1+\mathrm{i}\sin n\theta}{\cos\theta-1+\mathrm{i}\sin\theta}$.	**B1**
Multiplies numerator and denominator by complex conjugate.	**M1**
Takes real part: $\operatorname{Re}\left(\dfrac{z^n-1}{z-1}\right)=\dfrac{\cos n\theta\cos\theta+\sin n\theta\sin\theta-\cos n\theta-\cos\theta+1}{(\cos\theta-1)^2+\sin^2\theta}$.	**M1**
$=\dfrac{\cos n\theta\cos\theta+\sin n\theta\sin\theta-\cos n\theta}{2(1-\cos\theta)}+\frac{1}{2}$.	**A1**
Factorises: $=\dfrac{\cos n\theta(\cos\theta-1)+\sin n\theta\sin\theta}{2(1-\cos\theta)}+\frac{1}{2}$.	**M1**
Divides through by denominator (AG): $=\frac{1}{2}\left(1-\cos n\theta+\dfrac{\sin n\theta\sin\theta}{1-\cos\theta}\right)$.	**M1 A1**
(c)	
Forms sum of areas of rectangles given in the diagram (A0 if comparison with integral missing or unclear): $\int_0^1\cos x\,\mathrm{d}x<\frac{1}{n}+\frac{1}{n}\cos\frac{1}{n}+\frac{1}{n}\cos\frac{2}{n}+\ldots\frac{1}{n}\cos\frac{n-1}{n}$.	**M1 A1**
Applies result from part (b) with $\theta=\frac{1}{n}$ (AG): $=\frac{1}{n}\left(1+\cos\frac{1}{n}+\cos\frac{2}{n}+\cdots+\cos\frac{n-1}{n}\right)=\frac{1}{2n}\left(1-\cos\frac{n}{n}+\dfrac{\sin\frac{n}{n}\sin\frac{1}{n}}{1-\cos\frac{1}{n}}\right)$.	**M1 A1**
(d)	
Forms sum of areas of rectangles (A0 if comparison with integral missing or unclear): $\int_0^1\cos x\,\mathrm{d}x>\frac{1}{n}\cos\frac{1}{n}+\frac{1}{n}\cos\frac{2}{n}+\cdots+\frac{1}{n}\cos\frac{n}{n}$.	**M1 A1**
$=\dfrac{1}{2n}\left(\cos 1-1+\dfrac{\sin 1\sin\frac{1}{n}}{1-\cos\frac{1}{n}}\right)$.	**A1**
	Total: 15

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Q49

9231/21, 9231/22 • May/June 2024 • Question 4

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Mark scheme.

(a)	
Separates into correct structure: $I_n = \int_0^{\ln 3} \operatorname{sech}^{n-2} x \operatorname{sech}^2 x \, dx$.	**B1**
Uses integration by parts correctly: $I_n = \left[\operatorname{sech}^{n-2} x \tanh x\right]_0^{\ln 3} + (n-2)\int_0^{\ln 3} \operatorname{sech}^{n-2} x \tanh^2 x \, dx$.	***M1**
Uses $1 - \operatorname{sech}^2 x = \tanh^2 x$: $\left(\frac{3}{5}\right)^{n-2}\left(\frac{4}{5}\right) + (n-2)(I_{n-2} - I_n)$.	**DM1 A1**
AG: $(n-1)I_n = \left(\frac{3}{5}\right)^{n-2}\left(\frac{4}{5}\right) + (n-2)I_{n-2}$.	**A1**
(b)	
$I_2 = \frac{4}{5}$.	**B1**
Applies reduction formula with $n = 4$: $3I_4 = \left(\frac{3}{5}\right)^2\left(\frac{4}{5}\right) + 2I_2 \Rightarrow I_4 = \frac{236}{375} = 0.629$.	**M1 A1**
	Total: 8

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Q50

9231/21, 9231/22 • May/June 2024 • Question 5

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Mark scheme.

(a)	
Forms the sum of the areas of the rectangles, may be in summation form. M1 is for correct number of rectangles.	**M1 A1**
Applies formulae from MF19. Summations must have (correct) limits for A1. Substituted in correctly once for M1.	**M1 A1**
AG: $= 1 + \frac{1}{n} - \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \left(1+\frac{1}{n}\right)\left(\frac{2}{3}-\frac{1}{6n}\right)$. This A1 requires the previous A1.	**A1**
(b)	
Forms the sum of the areas of appropriate rectangles. M1 is for seeing correct height of last rectangle.	**M1 A1**
Applies formulae from MF19, substitutes correct limit. Substituted in correctly once for M1.	**M1**
$L_n = \left(1-\frac{1}{n}\right)\left(\frac{2}{3}+\frac{1}{6n}\right) = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$ or fully expanded or fully factorised.	**A1**
(c)	
Expresses their $U_n - L_n$ in terms of $\frac{1}{n}$. For M1 they must be taking the limit of an expression that tends to a constant. (Must be fully expanded if they do not take the limit at any stage.)	**M1**
AG, CWO: $\frac{1}{n} \to 0$ as $n \to \infty$. Their L_n must be correct.	**A1**
	Total: 11

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Q51

9231/23 • May/June 2024 • Question 1

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Mark scheme.

Completes the square: $-x^2+4x-1=3-(x-2)^2$.	**B1**
Applies formula: $\int_2^{7/2} \frac{1}{\sqrt{3-(x-2)^2}}\,\mathrm{d}x = \left[\sin^{-1}\left(\frac{x-2}{\sqrt{3}}\right)\right]_2^{7/2}$.	**M1 A1**
Uses limits. Answer must be in radians: $\sin^{-1}\left(\frac{1}{2}\sqrt{3}\right)-\sin^{-1}(0)=\frac{1}{3}\pi$.	**M1 A1**
	Total: 5

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Q52

9231/23 • May/June 2024 • Question 2

↑ Back to contents

Mark scheme.

(a)	
Differentiates x and y with respect to t: $\dot{x} = \sinh t$, $\dot{y} = \cosh t$.	**B1**
Applies $\cosh^2 t + \sinh^2 t = \cosh 2t$: $\dot{x}^2 + \dot{y}^2 = \sinh^2 t + \cosh^2 t = \cosh 2t$.	**M1 A1**
AG: $s = \int_0^{3/5} \sqrt{\cosh 2t}\,\mathrm{d}t$.	**A1**
(b)	
Finds first derivative: $\frac{\mathrm{d}}{\mathrm{d}t}(\sqrt{\cosh 2t}) = \sinh 2t(\cosh 2t)^{-1/2}$.	**B1**
Finds second derivative: $2\sqrt{\cosh 2t} - \sinh^2 2t(\cosh 2t)^{-3/2}$.	**B1**
Applies Maclaurin's series: $s = \int_0^{3/5} 1 + t^2\,\mathrm{d}t = \left[t + \frac{1}{3}t^3\right]_0^{3/5}$.	**M1 A1**
CWO: $= \frac{84}{125} = 0.672$.	**A1**
	Total: 9

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Q53

9231/23 • May/June 2024 • Question 4

↑ Back to contents

Mark scheme.

(a)

Forms sum of the areas of the rectangles. Need to see this sum written explicitly, may be in summation form, cannot be embedded in working: $2^{-2}+\cdots+(N-1)^{-2}=\sum_{r=2}^{N-1}\frac{1}{r^2}$.	**B1**
Evaluates integral, correct limits for M1: $\int_2^N x^{-2}\,\mathrm{d}x=\left[-x^{-1}\right]_2^N=\frac{1}{2}-\frac{1}{N}$.	**M1 A1**
Compares with integral correctly (using $>$): $\sum_{r=1}^{N} r^{-2}>\int_2^N x^{-2}\,\mathrm{d}x+1+\frac{1}{N^2}$.	**M1**
AG: $\sum_{r=1}^{N} r^{-2}>\frac{3}{2}-\frac{1}{N}+\frac{1}{N^2}$.	**A1**

(b)

Forms the sum of the areas of appropriate rectangles. Allow addition of one extra rectangle or shift one left or right if limits on the integral are adjusted accordingly: $3^{-2}+4^{-2}+\cdots+N^{-2}$.	**M1***
Compares with integral, correct limits and correct inequality symbol ($<$): $<\int_2^N x^{-2}\,\mathrm{d}x=\frac{1}{2}-\frac{1}{N}$.	**DM1**
$\sum_{r=1}^{N} r^{-2}<\frac{7}{4}-\frac{1}{N}$.	**A1**

(c)

Needs to clear (using $>$ or words) that the value is a lower bound: $\sum_{r=1}^{\infty} r^{-2}>\frac{3}{2}$.	**B1**
Needs to clear (using $<$ or words) that the value is an upper bound. Must have scored final A mark in part (b). FT on fully correct alternative to part (b): $\sum_{r=1}^{\infty} r^{-2}<\frac{7}{4}$.	**B1FT**

Total: 10

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Q54

9231/21, 9231/23 • October/November 2024 • Question 3

↑ Back to contents

Mark scheme.

(a)	
Correct formula and applies substitution: $A = 2\pi \int_{\ln 4/3}^{\ln 12/5} e^x \sqrt{1+e^{2x}}\,dx = 2\pi \int_{4/3}^{12/5} u\sqrt{1+u^2}\left(\frac{1}{u}\right) du$ (AG).	**M1A1**
(b)	
$u = \sinh v \Rightarrow du = \cosh v\,dv$.	**B1**
Applies substitution and $\cosh^2 x = 1 + \sinh^2 x$: $A = 2\pi \int_{\ln 3}^{\ln 5} \sqrt{1+\sinh^2 v}\cosh v\,dv = 2\pi \int_{\ln 3}^{\ln 5} \cosh^2 v\,dv$.	**M1A1**
Applies $2\cosh^2 x = \cosh 2x + 1$: $\pi \int_{\ln 3}^{\ln 5} (\cosh 2v + 1)\,dv$.	**M1**
Correct integration: $= \pi\left[\frac{1}{2}\sinh 2v + v\right]_{\ln 3}^{\ln 5}$.	**A1**
$\pi\left(\frac{312}{50} + \ln 5 - \frac{40}{18} - \ln 3\right) = \pi\left(\frac{904}{225} + \ln\frac{5}{3}\right)$ (AG).	**A1**
	Total: 8

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Q55

9231/21, 9231/23 • October/November 2024 • Question 6

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Mark scheme.

(a)	
Forms the sum of the areas of the rectangles (M1 for correct number of rectangles).	**M1A1**
Applies $\sum_{n=1}^{N} r^n = \frac{r(r^N-1)}{r-1}$ to obtain $L_N = \frac{1}{2N\left(2^{1/N}-1\right)}$ (AG).	**M1A1**
(b)	
Forms the sum of the areas of appropriate rectangles.	**M1A1**
Applies $\sum_{n=0}^{N-1} r^n = \frac{r^N-1}{r-1}$ to obtain $U_N = \frac{2^{1/N}}{2N\left(2^{1/N}-1\right)}$.	**M1A1**
(c)	
Simplifies $U_N - L_N$ to $\frac{c}{N}$: $\frac{2^{1/N}-1}{2N\left(2^{1/N}-1\right)} = \frac{1}{2N} \leq 10^{-3} \Rightarrow 2N \geq 10^3$.	**M1**
Least value of N is 500 (CWO).	**A1**
(d)	
Forms inequality $L_N < \frac{1}{2\ln 2} < U_N \Rightarrow \frac{1}{2U_N} < \ln 2 < \frac{1}{2L_N}$ (FT on their U_N).	**M1A1FT**
$0.693 = \frac{1}{2U_N} < \ln 2 < \frac{1}{2L_N} = 0.694$ (CWO, must have used $N = 500$).	**A1A1**
	Total: 14

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Q56

9231/22 • October/November 2024 • Question 1

↑ Back to contents

Mark scheme.

Applies formula: $\int_6^7 \frac{1}{\sqrt{(x-5)^2-1}}dx = \left[\cosh^{-1}(x-5)\right]_6^7$.	**M1A1**
Uses limits: $= \cosh^{-1}(2) - \cosh^{-1}(1)$.	**M1**
$= \ln(2+\sqrt{3})$.	**A1**
	Total: 4

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Q57

9231/22 • October/November 2024 • Question 3

↑ Back to contents

Mark scheme.

Differentiates x and y with respect to t: $\dot{x} = e^{2t} - t^2$, $\dot{y} = 2te^t$.	**B1B1**
Factorises $\dot{x}^2 + \dot{y}^2$: $\left(e^{2t} - t^2\right)^2 + 4t^2e^{2t} = \left(e^{2t} + t^2\right)^2$.	**M1A1**
Applies correct formula for arc length: $\int_0^1 e^{2t} + t^2 \, dt = \left[\frac{1}{2}e^{2t} + \frac{1}{3}t^3\right]_0^1$.	**M1A1**
$= \frac{1}{2}e^2 - \frac{1}{6}$.	**A1**
	Total: 7

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Q58

9231/22 • October/November 2024 • Question 6

↑ Back to contents

Mark scheme.

(a)	
Forms the sum of the areas of the rectangles (M1 for correct number of rectangles).	**M1A1**
Applies sum of geometric progression to get $U_n = \dfrac{e-1}{n(1-e^{-1/n})}$ (AG).	**M1A1**
(b)	
Forms the sum of the areas of appropriate rectangles.	**M1A1**
Applies sum of geometric progression: $L_n = \dfrac{e-1}{n(e^{1/n}-1)}$.	**M1A1**
(c)	
Simplifies $U_n - L_n = \dfrac{e-1}{n}$.	**M1**
$\dfrac{e-1}{n} \to 0$ as $n \to \infty$ (AG, CWO).	**A1**
(d)	
Substitutes power series for $e^{-1/z}$: $z(1 - e^{-1/z}) = z\left(1 - \left(1 - \frac{1}{z} + \frac{1}{2z^2} - \frac{1}{6z^3} + \dots\right)\right) = 1 - \frac{1}{2z} + \frac{1}{6z^2} + \dots$ (A1 for enough terms).	**M1A1**
$\lim\limits_{n\to\infty}(U_n) = e - 1$.	**B1**
	Total: 13

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Q59

9231/21, 9231/22 • May/June 2025 • Question 2

↑ Back to contents

Mark scheme.

(a)	
Applies integration by parts. Appropriate choice of parts for M1: $I_n = [(1-x)^n \cosh x]_0^1 + \int_0^1 n(1-x)^{n-1} \cosh x \, dx$.	**M1**
Correct result of first integration by parts.	**A1**
Applies integration by parts again. Appropriate choice of parts for M1: $I_n = -1 + n\left[(1-x)^{n-1} \sinh x\right]_0^1 + n(n-1)\int_0^1 (1-x)^{n-2} \sinh x \, dx$.	**M1**
$I_n = -1 + n(n-1)I_{n-2}$. AG, withhold for no evidence of substitution or incorrect use of limits.	**A1**
(b)	
Finds $I_0 = \int_0^1 \sinh x \, dx = [\cosh x]_0^1 = \cosh 1 - 1$.	**B1**
Uses reduction formula from (a): $I_2 = -1 + 2I_0$.	**M1**
$I_2 = 2\cosh 1 - 3$. Accept $e + e^{-1} - 3$.	**A1**
	Total: 7

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Q60

9231/21, 9231/22 • May/June 2025 • Question 4

↑ Back to contents

Mark scheme.

(a)

Forms sum of the areas of the rectangles: $\frac{1}{\sqrt{1}}e^{\sqrt{1}} + \frac{1}{\sqrt{2}}e^{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}}e^{\sqrt{n-1}}$.	**M1**
Compares with integral: $< \int_1^n \frac{1}{\sqrt{x}}e^{\sqrt{x}}\,dx$. Wrong comparison (missing/wrong inequality or wrong limits) is A0.	**A1**
Correct integration: $\int_1^n \frac{1}{\sqrt{x}}e^{\sqrt{x}}\,dx = \left[2e^{\sqrt{x}}\right]_1^n$. Attempt at integration by parts is M0.	**M1**
Correct result of integration.	**A1**
$\sum_{r=1}^{n} \frac{1}{\sqrt{r}}e^{\sqrt{r}} < \left(2 + \frac{1}{\sqrt{n}}\right)e^{\sqrt{n}} - 2e$. AG, must have obtained all previous marks.	**A1**

(b)

Forms the sum of the areas of appropriate rectangles: $\frac{1}{\sqrt{2}}e^{\sqrt{2}} + \frac{1}{\sqrt{3}}e^{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}e^{\sqrt{n}}$.	**M1**
Compares with integral: $> \int_1^n \frac{1}{\sqrt{x}}e^{\sqrt{x}}\,dx = \left[2e^{\sqrt{x}}\right]_1^n$.	**A1**
Adds term from $r = 1$.	**M1**
$\sum_{r=1}^{n} \frac{1}{\sqrt{r}}e^{\sqrt{r}} > \left[2e^{\sqrt{x}}\right]_1^n + e = 2e^{\sqrt{n}} - e$.	**A1**
	Total: 9

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Q61

9231/23 • May/June 2025 • Question 6

↑ Back to contents

Mark scheme.

(a)

Forms sum of the areas of the rectangles: $\frac{1}{n}\left(\frac{1}{\frac{1^2}{n^2}+1}+\frac{1}{\frac{2^2}{n^2}+1}+\cdots+\frac{1}{\frac{n^2}{n^2}+1}\right)=\frac{1}{n}\left(\frac{n^2}{1^2+n^2}+\frac{n^2}{2^2+n^2}+\cdots+\frac{n^2}{n^2+n^2}\right)$.	**M1 A1**
Evaluates integral: $\int_0^1 \frac{1}{x^2+1}\mathrm{d}x=\left[\tan^{-1}x\right]_0^1=\frac{1}{4}\pi$.	**M1**
CAO.	**A1**
$\frac{1}{n}\sum_{r=1}^{n}\frac{n^2}{n^2+r^2}=\sum_{r=1}^{n}\frac{n}{n^2+r^2}<\frac{1}{4}\pi$. AG. Clear and convincing comparison with the integral.	**A1**

(b)

Forms the sum of the areas of appropriate rectangles: $\frac{1}{n}\left(\frac{1}{0^2+1}+\frac{1}{\frac{1^2}{n^2}+1}+\cdots+\frac{1}{\frac{(n-1)^2}{n^2}+1}\right)=\frac{1}{n}\left(\frac{n^2}{0^2+n^2}+\frac{n^2}{1^2+n^2}+\cdots+\frac{n^2}{(n-1)^2+n^2}\right)$. M0 for more than n rectangles.	***M1 A1**
Compares with integral, correctly adjusting for 0^{th} and last rectangles (allow sign errors only for M1; limits must match area covered by chosen rectangles): $\frac{1}{n}+\sum_{r=1}^{n}\frac{n}{n^2+r^2}-\frac{1}{2n}\geqslant\frac{1}{4}\pi\Rightarrow\sum_{r=1}^{n}\frac{n}{n^2+r^2}\geqslant\frac{1}{4}\pi-\frac{1}{2n}$.	**DM1 A1**

(c)

$\frac{1}{4}\pi$.	**B1**
	Total: 10

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Q62

9231/24 • May/June 2025 • Question 2

↑ Back to contents

Mark scheme.

Completes the square, OE: $x^2 - 2x + 5 = 4 + (x-1)^2$.	**B1**
Applies formula: $\int_1^{5/2} \frac{1}{\sqrt{4+(x-1)^2}}\,\mathrm{d}x = \left[\sinh^{-1}\left(\frac{x-1}{2}\right)\right]_1^{5/2}$.	**M1 A1**
Uses limits: $= \sinh^{-1}\left(\frac{3}{4}\right) - \sinh^{-1}(0)$.	**M1**
Uses logarithmic form: $= \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right) = \ln 2$.	**M1 A1**
	Total: 6

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Q63

9231/21, 9231/23 • October/November 2025 • Question 3

↑ Back to contents

Mark scheme.

Uses the product rule: $\dfrac{d}{dx}\left(x(1+x^2)^n\right) = 2nx^2(1+x^2)^{n-1} + (1+x^2)^n$.	**(a) M1**
Correct differentiation.	**A1**
Uses $x^2 = x^2 + 1 - 1$: $= 2n(1+x^2-1)(1+x^2)^{n-1} + (1+x^2)^n$.	**M1***
Integrates both sides using the limits given: $\left[x(1+x^2)^n\right]_0^1 = 2nI_n - 2nI_{n-1} + I_n$.	**M1 (dep)**
Substitutes limits and rearranges: $2^n = (2n+1)I_n - 2nI_{n-1} \Rightarrow (2n+1)I_n = 2^n + 2nI_{n-1}$. AG.	**A1**
$I_{-1} = \left[\tan^{-1} x\right]_0^1 = \frac{1}{4}\pi$.	**(b) B1**
Applies reduction formula with $n = -1$: $-I_{-1} = \frac{1}{2} - 2I_{-2}$.	**M1**
Correct substitution.	**A1**
$-\frac{1}{4}\pi = \frac{1}{2} - 2I_{-2} \Rightarrow I_{-2} = \frac{1}{4} + \frac{1}{8}\pi$.	**A1**
	Total: 9

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Q64

9231/21, 9231/23 • October/November 2025 • Question 5

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Mark scheme.

Forms the sum of the areas of the rectangles $\left[\int_0^1 \left(\frac{1}{3}x^3+x\right)dx <\right] \left(\frac{1}{n}\right)\left(\frac{1}{3}\left(\frac{1}{n}\right)^3+\frac{1}{n}\right)+\cdots+\left(\frac{1}{n}\right)\left(\frac{1}{3}\left(\frac{n}{n}\right)^3+\frac{n}{n}\right)$. M1 for correct width and at least three complete terms including first and last.	**(a) M1**
Correct sum.	**A1**
Applies formulae from MF19: $\frac{1}{3n^4}\sum_{r=1}^{n} r^3+\frac{1}{n^2}\sum_{r=1}^{n} r=\frac{n^2(n+1)^2}{12n^4}+\frac{n(n+1)}{2n^2}$.	**M1**
Correct application.	**A1**
$=\frac{1}{12}(1+\frac{1}{n})^2+\frac{1}{2}(1+\frac{1}{n})=\frac{1}{12}(1+\frac{1}{n})(7+\frac{1}{n})$. AG.	**A1**
Forms the sum of the areas of appropriate rectangles (using left endpoints).	**(b) M1**
Correct sum.	**A1**
Applies formulae from MF19: $\frac{1}{3n^4}\sum_{r=1}^{n-1} r^3+\frac{1}{n^2}\sum_{r=1}^{n-1} r=\frac{(n-1)^2n^2}{12n^4}+\frac{(n-1)n}{2n^2}$. Accept $L_n=U_n-\frac{4}{3n}$.	**M1**
$=\frac{1}{12}(1-\frac{1}{n})^2+\frac{1}{2}(1-\frac{1}{n})=\frac{1}{12}(1-\frac{1}{n})(7-\frac{1}{n})$. At least one line of simplification.	**A1**
Simplifies U_n-L_n to $\frac{c}{n}$: $U_n-L_n=\frac{1}{12}\left(\frac{16}{n}\right)=\frac{4}{3n}$.	**(c) M1**
$\frac{4}{3n}\to 0$ as $n\to\infty$. AG.	**A1**
	Total: 11

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Q65

9231/22 • October/November 2025 • Question 1

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Mark scheme.

Forms length of C correctly: $s = \int_0^\pi \sqrt{e^{\frac{8}{3}\theta} + \frac{16}{9}e^{\frac{8}{3}\theta}}\, d\theta = \frac{5}{3}\int_0^\pi e^{\frac{4}{3}\theta}\, d\theta$.	**M1**
Correct setup.	**A1**
Integrates an expression of the form $ae^{b\theta}$. One error in a or b allowed: $s = \frac{5}{4}\left[e^{\frac{4}{3}\theta}\right]_0^\pi$.	**M1**
$s = \frac{5}{4}\left(e^{\frac{4}{3}\pi} - 1\right)$. Must be exact.	**A1**
	Total: 4

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Q66

9231/22 • October/November 2025 • Question 4(b)

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Mark scheme.

Applies part (a): $\int_0^{\frac{1}{2}\pi} \cos^6 2x \,\mathrm{d}x = \frac{1}{2}\int_0^{\pi} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right) \mathrm{d}\theta$. Must have the $\frac{1}{2}$ and correct limits.	**M1**
$= \frac{1}{2}\left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\pi}$.	**A1**
$= \frac{5}{32}\pi$.	**A1**
	Total: 3

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Q67

9231/22 • October/November 2025 • Question 6

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Mark scheme.

Applies substitution and $\sinh^2 u + 1 = \cosh^2 u$: $\frac{dx}{du} = \frac{1}{2}\sqrt{2}\cosh u \Rightarrow \int \frac{1}{\sqrt{2x^2+1}}dx = \frac{1}{2}\sqrt{2}\int 1\, du$.	**(a) M1 A1**
$= \frac{1}{2}\sqrt{2}\sinh^{-1}(x\sqrt{2})[+C]$; equivalently $\sinh^{-1}(x\sqrt{2}) = \ln(x\sqrt{2} + \sqrt{2x^2+1})$.	**A1**
Forms sum of the areas of the rectangles.	**(b) M1* A1**
Compares with integral: $< \int_0^1 \frac{1}{\sqrt{2x^2+1}}dx$.	**M1 (dep)**
Evaluates integral: $\int_0^1 \frac{1}{\sqrt{2x^2+1}}dx = \left[\frac{1}{2}\sqrt{2}\sinh^{-1}(x\sqrt{2})\right]_0^1 = \frac{1}{2}\sqrt{2}\ln(\sqrt{2}+\sqrt{3})$.	**B1**
AG, must see $\frac{1}{n}\sum_{r=1}^{n} \frac{n}{\sqrt{2r^2+n^2}} = \sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}} < \frac{1}{2}\sqrt{2}\ln(\sqrt{2}+\sqrt{3})$.	**A1**
Forms the sum of the areas of appropriate rectangles.	**(c) M1* A1**
Compares with integral.	**M1 (dep)**
$\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}} \geq \frac{1}{2}\sqrt{2}\ln(\sqrt{2}+\sqrt{3}) + \frac{1}{n}\left(\frac{1}{3}\sqrt{3} - 1\right)$.	**A1**
$\frac{1}{2}\sqrt{2}\ln(\sqrt{2}+\sqrt{3})$.	**(d) B1**
	Total: 13

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Q68

9231/24 • October/November 2025 • Question 1

↑ Back to contents

Mark scheme.

Integrates by parts. Must see first term $x^n \cosh x$: $I_n = [x^n \cosh x]_0^1 - n\int_0^1 x^{n-1}\cosh x \, dx$.	**(a) M1**
Integrates by parts again: $= [x^n \cosh x]_0^1 - n\left([x^{n-1}\sinh x]_0^1 - (n-1)\int_0^1 x^{n-2}\sinh x \, dx\right)$.	**M1**
$I_n = \cosh 1 - n(\sinh 1 - (n-1)I_{n-2}) = \cosh 1 - n\sinh 1 + n(n-1)I_{n-2}$. AG. Omission of brackets or limits is A0.	**A1**
Applies reduction formula: $I_2 = \cosh 1 - 2\sinh 1 + 2I_0$.	**(b) M1**
$I_2 = 3\cosh 1 - 2\sinh 1 - 2$. Accept $\frac{1}{2}(e + 5e^{-1}) - 2$.	**A1**
	Total: 5

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Q69

9231/24 • October/November 2025 • Question 4

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Mark scheme.

Forms the sum of the areas of the rectangles. M1 for at least three terms including first and last, and common ratio $3^{1/N}$. Missing "+" is A0.	**(a) M1** **A1**
Applies $\sum_{n=1}^{N} r^n = \dfrac{r(r^N-1)}{r-1}$: $= \dfrac{1}{2N}\sum_{n=1}^{N}(3^{1/N})^n = \dfrac{3^{1/N}}{N(3^{1/N}-1)}$. AG.	**M1 A1**
Forms the sum of the areas of appropriate rectangles. M1 for at least three terms including first and last.	**(b) M1** **A1**
Applies $\sum_{n=0}^{N-1} r^n = \dfrac{r^N-1}{r-1}$: $= \dfrac{1}{2N}\sum_{n=0}^{N-1}(3^{1/N})^n = \dfrac{1}{N(3^{1/N}-1)}$.	**M1 A1**
Simplifies their $U_N - L_N$ to $\dfrac{c}{N}$: $\dfrac{3^{1/N}}{N(3^{1/N}-1)} - \dfrac{1}{N(3^{1/N}-1)} = \dfrac{1}{N}$.	**(c) M1**
$\dfrac{1}{N} \to 0$ as $N \to \infty$.	**A1**
	Total: 10

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Q70

9231/24 • October/November 2025 • Question 7(a)

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Mark scheme.

Differentiates x with respect to t: $\dot{x} = 4\frac{\sec^2 \frac{1}{2}t}{\tan \frac{1}{2}t} + 3\operatorname{cosec}^2 t - 3 = 8\operatorname{cosec} t + 3\operatorname{cosec}^2 t - 3 = 8\operatorname{cosec} t + 3\cot^2 t$. AG.	**(i) B1**
Differentiates y with respect to t: $\dot{y} = 6\operatorname{cosec} t - 4\cot^2 t$.	**(ii) B1**
Expands and simplifies $\dot{x}^2 + \dot{y}^2$. A1 for reaching $100\operatorname{cosec}^2 t + 25\cot^4 t$.	**(iii) M1 A1**
Factorises $\dot{x}^2 + \dot{y}^2 = 25(\operatorname{cosec}^2 t + 1)^2$ (ALT: $25(\cot^2 t + 2)^2$).	**M1* A1**
Applies correct formula for arc length and integrates: $5\int_{\pi/6}^{\pi/3}(\operatorname{cosec}^2 t + 1)\mathrm{d}t = 5[-\cot t + t]_{\pi/6}^{\pi/3} = \frac{5}{6}\pi + \frac{10}{3}\sqrt{3}$. Answer in terms of π and $\sqrt{3}$ for A1.	**M1 (dep) A1**
	Total: 8

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Written by Ali Hashir.

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