Further Math with Ali Hashir

Hyperbolic Functions

Topical Questions (CIE 9231)

Compiled from CIE 9231 Further Mathematics past papers (2015–2025), sorted by topic. Each question is numbered for this booklet, with its original paper code and session shown beneath.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Q1

9231/21, 9231/22 • May/June 2020 • Question 5

The curves $C_1 : y = \cosh x$ and $C_2 : y = \sinh 2x$ intersect at the point where $x = a$.

(a) Find the exact value of a, giving your answer in logarithmic form. [4]

(b) Sketch C_1 and C_2 on the same diagram. [2]

(c) Find the exact value of the length of the arc of C_1 from $x = 0$ to $x = a$. [5]

Q2

9231/23 • May/June 2020 • Question 6

(a) Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1 - \tanh^2\theta = \operatorname{sech}^2\theta.$$

[3]

The variables x and y are such that $\tanh y = \cos\left(x + \frac{1}{4}\pi\right)$, for $-\frac{1}{4}\pi < x < \frac{3}{4}\pi$.

(b) By differentiating the equation $\tanh y = \cos\left(x + \frac{1}{4}\pi\right)$ with respect to x, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{cosec}\left(x + \tfrac{1}{4}\pi\right).$$

[4]

(c) Hence find the first three terms in the Maclaurin's series for $\tanh^{-1}\left(\cos(x + \frac{1}{4}\pi)\right)$ in the form $\frac{1}{2}\ln a + bx + cx^2$, giving the exact values of the constants a, b and c. [5]

Q3

9231/21, 9231/23 • October/November 2020 • Question 8

(a) Sketch the graph of $y = \coth x$ for $x > 0$ and state the equations of the asymptotes. [2]

(b) Starting from the definitions of coth and cosech in terms of exponentials, prove that

$$\coth^2 x - \operatorname{cosech}^2 x = 1.$$

[3]

The curve C has equation $y = \ln \coth\left(\frac{1}{2}x\right)$ for $x > 0$.

(c) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{cosech} x$. [3]

(d) It is given that the arc length of C from $x = a$ to $x = 2a$ is $\ln 4$, where a is a positive constant. Show that $\cosh a = 2$ and find, in logarithmic form, the exact value of a. [7]

Q4

9231/22 • October/November 2020 • Question 2

A curve has equation $y = \cosh x$, for $0 \leqslant x \leqslant \frac{1}{2}$.

Find, in terms of π and e, the area of the surface generated when the curve is rotated through 2π radians about the x-axis. [6]

Q5

9231/21, 9231/22 • May/June 2021 • Question 7

(a) It is given that $y = \text{sech}^{-1}\left(x + \frac{1}{2}\right)$.

Express $\cosh y$ in terms of x and hence show that $\sinh y \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{\left(x + \frac{1}{2}\right)^2}$. [3]

(b) Find the first three terms in the Maclaurin's series for $\text{sech}^{-1}\left(x + \frac{1}{2}\right)$ in the form

$$\ln a + bx + cx^2,$$

where a, b and c are constants to be determined. [7]

Q6

9231/21, 9231/22 • May/June 2021 • Question 8

The curve C has parametric equations

$$x = 2\cosh t, \qquad y = \tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t, \qquad \text{for } 0 \leq t \leq 1.$$

(a) Find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ and show that $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 - \sinh^2 t$. [3]

The area of the surface generated when C is rotated through 2π radians about the x-axis is denoted by A.

(b) (i) Show that $A = \pi \displaystyle\int_0^1 \left(\tfrac{3}{2}t - \tfrac{1}{4}\sinh 2t\right)(1 + \cosh 2t)\,\mathrm{d}t$. [4]

(ii) Hence find A in terms of π, $\sinh 2$ and $\cosh 2$. [6]

Q7

9231/23 • May/June 2021 • Question 2

Find the Maclaurin's series for $\ln\cosh x$ up to and including the term in x^4. [7]

Q8

9231/21, 9231/23 • October/November 2021 • Question 8

(a) Starting from the definition of cosh in terms of exponentials, prove that

$$2\cosh^2 A = \cosh 2A + 1.$$

[3]

The curve C has parametric equations

$$x = 2\cosh 2t + 3t, \qquad y = \tfrac{3}{2}\cosh 2t - 4t, \quad \text{for } -\tfrac{1}{2} \leq t \leq \tfrac{1}{2}.$$

The area of the surface generated when C is rotated through 2π radians about the y-axis is denoted by A.

(b) (i) Show that $A = 10\pi \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cosh 2t + 3t)\cosh 2t \, \mathrm{d}t$. [4]

(ii) Hence find A in terms of π and e. [7]

Q9

9231/22 • October/November 2021 • Question 8(a)(b)

(a) Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1 - \tanh^2 x = \operatorname{sech}^2 x.$$

[3]

(b) Using the substitution $u = \tanh x$, or otherwise, find $\int \operatorname{sech}^2 x \tanh^2 x \, \mathrm{d}x$. [2]

Q10

9231/21, 9231/22 • May/June 2022 • Question 2

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\cosh 2x = 2\sinh^2 x + 1.$$

[3]

(b) Find the set of values of k for which $\cosh 2x = k \sinh x$ has two distinct real roots. [5]

Q11

9231/21, 9231/23 • October/November 2022 • Question 4

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$\cosh^2 x - \sinh^2 x = 1.$$

[3]

(b) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}(\sinh x)\right) = \operatorname{sech} x$. [3]

(c) Sketch the graph of $y = \operatorname{sech} x$, stating the equation of the asymptote. [2]

(d) By considering a suitable set of n rectangles of unit width, use your sketch to show that

$$\sum_{r=1}^{n} \operatorname{sech} r < \tan^{-1}(\sinh n).$$

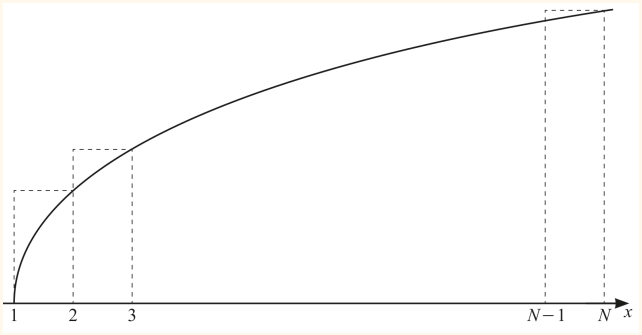
[3]

(e) Hence state an upper bound, in terms of π, for $\displaystyle\sum_{r=1}^{\infty} \operatorname{sech} r$. [1]

Q12

9231/21, 9231/22 • May/June 2023 • Question 7

(a) Use the substitution $u = x^2 - 1$ to find $\displaystyle\int \frac{x}{\sqrt{x^2-1}}\,\mathrm{d}x$. [3]



The diagram shows the curve with equation $y = \cosh^{-1} x$ together with a set of $(N-1)$ rectangles of unit width.

(b) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=2}^{N} \ln\left(r + \sqrt{r^2-1}\right) > N \ln\left(N + \sqrt{N^2-1}\right) - \sqrt{N^2-1}.$$

[5]

(c) Use a similar method to find, in terms of N, an upper bound for $\displaystyle\sum_{r=2}^{N} \ln\left(r + \sqrt{r^2-1}\right)$.

[3]

Q13

9231/21, 9231/22 • May/June 2023 • Question 8

(a) Starting from the definitions of sech and tanh in terms of exponentials, prove that

$$1 - \operatorname{sech}^2 t = \tanh^2 t.$$

[3]

The curve C has parametric equations

$$x = \tfrac{1}{2}\tanh^2 t + \ln \operatorname{sech} t, \qquad y = 1 + \tanh^4 t, \qquad \text{for } t > 0.$$

(b) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -4\operatorname{sech}^2 t.$ [5]

(c) Find the coordinates of the point on C with $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\dfrac{9}{2}$, giving your answer in the form $(a + \ln b, c)$ where a, b and c are rational numbers. [6]

Q14

9231/23 • May/June 2023 • Question 5

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$2\cosh^2 x = \cosh 2x + 1.$$

[3]

(b) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\tanh x = 1$$

for which $y = 1$ when $x = 0$. Give your answer in the form $y = \mathrm{f}(x)$. [8]

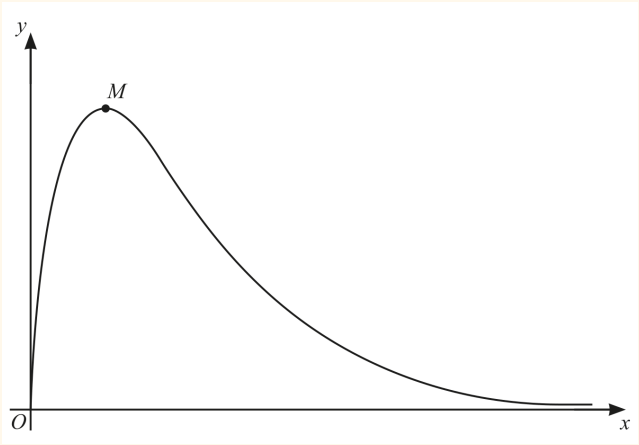
Q15

9231/21, 9231/23 • October/November 2023 • Question 3

Find the first three terms in the Maclaurin's series for $\tanh^{-1}\left(\frac{1}{2}e^{x}\right)$ in the form $\frac{1}{2}\ln a + bx + cx^2$, giving the exact values of the constants a, b and c. [6]

Q16

9231/22 • October/November 2023 • Question 5



The diagram shows part of the curve $y = x\,\mathrm{sech}^2 x$ and its maximum point M.

(a) Show that, at M,

$$2x \tanh x - 1 = 0$$

and verify that this equation has a root between 0.7 and 0.8. [4]

(b) By considering a suitable set of rectangles, use the diagram to show that

$$\sum_{r=2}^{n} r\,\mathrm{sech}^2 r < n \tanh n + \ln \mathrm{sech}\, n - \tanh 1 - \ln \mathrm{sech}\, 1.$$

[6]

Q17

9231/22 • October/November 2023 • Question 7

(a) Starting from the definitions of cosh and sinh in terms of exponentials, prove that

$$2\sinh^2 A = \cosh 2A - 1.$$

[3]

(b) A curve has equation $y = x^2$, for $0 \leqslant x \leqslant \frac{2}{3}$. The area of the surface generated when the curve is rotated through 2π radians about the x-axis is denoted by S.

Use the substitution $x = \frac{1}{2}\sinh u$ to show that $S = \frac{1}{32}\pi\left(\frac{820}{81} - \ln 3\right)$. [9]

Q18

9231/21, 9231/22 • May/June 2025 • Question 6

(a) Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1 - \tanh^2 u = \operatorname{sech}^2 u.$$

[3]

(b) Show that $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\operatorname{sech}^{-1} t\right) = -\dfrac{1}{t\sqrt{1-t^2}}$. [4]

It is given that

$$x = \tanh^{-1} t \quad \text{and} \quad y = t \operatorname{sech}^{-1} t, \quad \text{for } 0 < t < 1.$$

(c) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{1-t^2} + (1-t^2)\operatorname{sech}^{-1} t$. [4]

(d) Find $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ in terms of t. [4]

Q19

9231/23 • May/June 2025 • Question 2

(a) Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$\tanh^2 t + \operatorname{sech}^2 t = 1.$$

[3]

(b) The curve C has parametric equations

$$x = \ln(\cosh t), \quad y = \tan^{-1}(\sinh t), \quad \text{for } 0 \leqslant t \leqslant 1.$$

Find the length of C. [5]

Q20

9231/24 • May/June 2025 • Question 8

The curve C has equation $y = \tanh x$ for $x \geqslant 0$.

(a) Sketch C and state the equation of the asymptote. [2]

(b) By considering a suitable set of N rectangles of unit width, use your sketch to show that

$$\sum_{r=1}^{N} \tanh r > \ln(\cosh N).$$

[3]

(c) The arc of C joining the point where $x = 0$ to the point where $x = \frac{1}{2}\ln 3$ is rotated through one complete revolution about the x-axis. The area of the surface generated is denoted by S.

(i) Use the substitution $u = \sqrt{1 + \operatorname{sech}^4 x}$ to show that

$$S = \pi \int_{5/4}^{\sqrt{2}} \frac{u^2}{u^2 - 1}\, \mathrm{d}u.$$

[7]

(ii) Find the exact value of $\pi \displaystyle\int_{5/4}^{\sqrt{2}} \frac{u^2}{u^2 - 1}\, \mathrm{d}u$. You need not simplify your answer. [3]

Q21

9231/21, 9231/23 • October/November 2025 • Question 2

The curve C has parametric equations

$$x = \sinh t \quad \text{and} \quad y = t + \cosh t.$$

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [2]

(b) Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{1 - \sinh t}{\cosh^3 t}$. [4]

(c) Find the Maclaurin's series for y in terms of x up to and including the term in x^2. [2]

Q22

9231/21, 9231/23 • October/November 2025 • Question 7(a)

Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2}$. [3]

Q23

9231/24 • October/November 2025 • Question 2

(a) Find the exact values of x for which $\cosh 2x = 6\sinh^2 x$, giving your answers in logarithmic form. [4]

(b) Sketch the curves $C_1 : y = \cosh 2x$ and $C_2 : y = 6\sinh^2 x$ on the same diagram. [2]

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

Written by Ali Hashir.

Answers available in the companion mark scheme booklet.