Further Math with Ali Hashir

Hyperbolic Functions

Topical Mark Schemes (CIE 9231)

Mark schemes corresponding to the topical questions booklet for this topic, compiled from official CIE 9231 marking schemes (2015–2025). Question numbers match the questions booklet.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Contents

Click a question number below to jump straight to its mark scheme. Every box is a live link.

For the best hyperlink experience, open this PDF in a dedicated reader – such as Adobe Acrobat, Apple Preview, or your browser's built-in PDF viewer – rather than a basic in-app or embedded preview, where links sometimes don't work.

Q1

9231/21, 9231/22 • May/June 2020 • Question 5

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Mark scheme.

(a)	
$\cosh a = 2\sinh a \cosh a \Rightarrow \sinh a = \frac{1}{2}$	**M1 A1**
$a = \sinh^{-1}\frac{1}{2} = \ln\left(\frac{1}{2} + \sqrt{\frac{1}{4}+1}\right)$	**M1**
$a = \ln\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)$	**A1**
(b)	
C_1 correct	**B1**
C_2 correct and intersecting C_1 in the first quadrant	**B1**
(c)	
$\int_0^a \sqrt{1+\sinh^2 x}\,\mathrm{d}x$	**M1**
$\int_0^a \sqrt{\cosh^2 x}\,\mathrm{d}x = \int_0^a \cosh x\,\mathrm{d}x$	**M1 A1**
$[\sinh x]_0^a = \sinh a$	**M1**
$\frac{1}{2}$	**A1**
	Total: 11

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Q2

9231/23 • May/June 2020 • Question 6

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Mark scheme.

(a)

$\tanh\theta = \dfrac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}$, $\operatorname{sech}\theta = \dfrac{2}{e^{\theta}+e^{-\theta}}$	**B1**
$1-\left(\dfrac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}\right)^2 = \dfrac{(e^{\theta}+e^{-\theta})^2-(e^{\theta}-e^{-\theta})^2}{(e^{\theta}+e^{-\theta})^2} = \dfrac{4}{(e^{\theta}+e^{-\theta})^2}$	**M1**
$= \operatorname{sech}^2\theta$ (AG)	**A1**

(b)

$\tanh y = \cos(x+\frac{1}{4}\pi) \Rightarrow \operatorname{sech}^2 y\,\dfrac{dy}{dx} = -\sin(x+\frac{1}{4}\pi)$	**M1 A1**
$\left(1-\cos^2(x+\frac{1}{4}\pi)\right)\dfrac{dy}{dx} = -\sin(x+\frac{1}{4}\pi)$	**M1**
$\dfrac{dy}{dx} = -\dfrac{\sin(x+\frac{1}{4}\pi)}{\sin^2(x+\frac{1}{4}\pi)} = -\operatorname{cosec}(x+\frac{1}{4}\pi)$ (AG)	**A1**

(c)

$\dfrac{d^2y}{dx^2} = \cot(x+\frac{1}{4}\pi)\operatorname{cosec}(x+\frac{1}{4}\pi)$	**B1**
$y'(0) = -\operatorname{cosec}(\frac{1}{4}\pi)$, $y''(0) = \cot(\frac{1}{4}\pi)\operatorname{cosec}(\frac{1}{4}\pi)$	**M1**
$y(0) = \tanh^{-1}\left(\frac{1}{2}\sqrt{2}\right) = \frac{1}{2}\ln\left(\dfrac{2+\sqrt{2}}{2-\sqrt{2}}\right)$	**M1**
$y = \frac{1}{2}\ln(3+2\sqrt{2}) - x\sqrt{2} + \frac{1}{2}x^2\sqrt{2}$	**M1 A1**
	Total: 12

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Q3

9231/21, 9231/23 • October/November 2020 • Question 8

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Mark scheme.

(a) Correct shape and position, not too truncated.	**B1**
States equations of asymptotes: $x = 0$, $y = 1$.	**B1**
(b) $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$, $\operatorname{cosech} x = \dfrac{2}{e^x - e^{-x}}$.	**B1**
Writes over common denominator, AG: $\left(\dfrac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - \dfrac{4}{(e^x - e^{-x})^2} = \dfrac{e^{2x} + e^{-2x} - 2}{(e^x - e^{-x})^2} = 1$.	**M1 A1**
(c) Uses chain rule: $\dfrac{dy}{dx} = -\dfrac{\operatorname{sech}^2\left(\frac{1}{2}x\right)}{2\tanh\left(\frac{1}{2}x\right)} = -\dfrac{1}{2\sinh\left(\frac{1}{2}x\right)\cosh\left(\frac{1}{2}x\right)}$.	**M1 A1**
AG: $= -\dfrac{1}{\sinh(x)} = -\operatorname{cosech} x$.	**A1**
(d) Forms correct integral: $\displaystyle\int_a^{2a} \sqrt{1 + \operatorname{cosech}^2 x}\, dx$.	**M1**
Uses $\coth^2 x - \operatorname{cosech}^2 x = 1$: $\displaystyle\int_a^{2a} \sqrt{\coth^2 x}\, dx = \int_a^{2a} \coth x\, dx$.	**M1 A1**
Integrates and substitutes limits: $= [\ln \sinh x]_a^{2a} = \ln \sinh 2a - \ln \sinh a$.	**M1**
Combines logarithms and uses double angle formula: $= \ln \dfrac{\sinh 2a}{\sinh a} = \ln(2\cosh a)$.	**M1**
AG: $\ln(2\cosh a) = \ln 4 \Rightarrow \cosh a = 2$.	**A1**
Must reject $\ln\left(2 - \sqrt{3}\right)$: $a = \ln\left(2 + \sqrt{2^2 - 1}\right) = \ln\left(2 + \sqrt{3}\right)$.	**A1**
	Total: 15

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Q4

9231/22 • October/November 2020 • Question 2

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Mark scheme.

Applies $\cosh^2 x = 1 + \sinh^2 x$: $\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1 + \sinh^2 x} = \cosh x$.	**M1 A1**
Correct formula, correct limits: $2\pi \int_0^{\frac{1}{2}} \cosh^2 x \,\mathrm{d}x$.	**M1**
Applies $2\cosh^2 x = \cosh 2x + 1$ or expands $2(\mathrm{e}^x + \mathrm{e}^{-x})^2$: $= \pi \int_0^{\frac{1}{2}} \cosh 2x + 1 \,\mathrm{d}x$.	**M1**
Correct integration: $= \pi \left[\frac{1}{2}\sinh 2x + x\right]_0^{\frac{1}{2}}$.	**A1**
$= \frac{1}{4}\pi \left(\mathrm{e} - \mathrm{e}^{-1} + 2\right)$.	**A1**
	Total: 6

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Q5

9231/21, 9231/22 • May/June 2021 • Question 7

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Mark scheme.

(a)

Relates to $\cosh y$: $\operatorname{sech} y = \dfrac{1}{\cosh y} = x + \frac{1}{2} \Rightarrow \cosh y = \left(x + \frac{1}{2}\right)^{-1}$.	**B1**
Differentiates LHS: $\sinh y \dfrac{dy}{dx}$.	**B1**
Differentiates RHS (AG): $-\left(x + \frac{1}{2}\right)^{-2}$.	**B1**

(b)

$\sinh \dfrac{d^2y}{dx^2} + \cosh y \left(\dfrac{dy}{dx}\right)^2 = 2\left(x + \frac{1}{2}\right)^{-3}$ (M1 A1 for LHS; B1 for RHS).	**M1 A1 B1**
Relates to $\cosh^{-1}$ and uses logarithmic form: $y(0) = \operatorname{sech}^{-1}\left(\frac{1}{2}\right) = \cosh^{-1}(2) = \ln\left(2 + \sqrt{3}\right)$.	**M1 A1**
Evaluates derivatives at $x = 0$: $y'(0) = -\dfrac{4}{\sqrt{3}}$, $y''(0) = \dfrac{16}{3\sqrt{3}}$.	**M1**
$y = \ln\left(2 + \sqrt{3}\right) - \dfrac{4}{\sqrt{3}}x + \dfrac{8}{3\sqrt{3}}x^2$.	**A1**
	Total: 10

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Q6

9231/21, 9231/22 • May/June 2021 • Question 8

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Mark scheme.

(a)	
$\dfrac{dx}{dt} = 2\sinh t$.	**B1**
Applies $2\sinh^2 t = \cosh 2t - 1$: $\dfrac{dy}{dt} = \frac{3}{2} - \frac{1}{2}\cosh 2t = 1 - \left(\frac{1}{2}\cosh 2t - \frac{1}{2}\right) = 1 - \sinh^2 t$ (AG).	**M1 A1**
(b)(i)	
Factorises $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = 4s^2 + (s^2-1)^2 = 4s^2 + s^4 - 2s^2 + 1 = (s^2+1)^2$.	**M1**
$= \cosh^4 t$.	**A1**
Correct formula for surface area (AG; A0 if limits missing): $2\pi\int_0^1 \left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right)\cosh^2 t\,dt = \pi\int_0^1 \left(\frac{3}{2}t - \frac{1}{4}\sinh 2t\right)(\cosh 2t + 1)\,dt$.	**M1 A1**
(b)(ii)	
Integrates: $\int_0^1 \frac{1}{4}\sinh 2t(\cosh 2t + 1)\,dt = \left[\frac{1}{16}(1+\cosh 2t)^2\right]_0^1 = \frac{1}{16}(1+\cosh 2)^2 - \frac{1}{4}$.	**M1 A1**
Integrates by parts: $\frac{3}{2}\int_0^1 t(\cosh 2t + 1)\,dt = \frac{3}{2}\left[t\left(\frac{1}{2}\sinh 2t + t\right)\right]_0^1 - \frac{3}{2}\int_0^1 \frac{1}{2}\sinh 2t + t\,dt$.	**M1 A1**
$= \frac{3}{2}\left(\frac{1}{2}\sinh 2 - \frac{1}{4}\cosh 2 + \frac{3}{4}\right)$.	**A1**
$\pi\left(\frac{3}{4}\sinh 2 - \frac{3}{8}\cosh 2 - \frac{1}{16}(1+\cosh 2)^2 + \frac{11}{8}\right)$ (OE, must be exact; decimal answer is 3.980131435 . . .).	**A1**
	Total: 13

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Q7

9231/23 • May/June 2021 • Question 2

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Mark scheme.

Finds first derivative: $\dfrac{dy}{dx} = \tanh x$.	**B1**
Finds second derivative: $\dfrac{d^2y}{dx^2} = \operatorname{sech}^2 x$.	**B1**
Finds third derivative: $\dfrac{d^3y}{dx^3} = -2\tanh x \operatorname{sech}^2 x = -2\dfrac{\sinh x}{\cosh^3 x}$.	**M1 A1**
Finds fourth derivative: $\dfrac{d^4y}{dx^4} = -2\dfrac{\cosh^2 x - 3\sinh^2 x}{\cosh^4 x} = -2\operatorname{sech}^4 x + 4\tanh^2 x \operatorname{sech}^2 x$ (alternative: $6\tanh^2 x \operatorname{sech}^2 x - 2\operatorname{sech}^2 x$).	**B1**
Evaluates derivatives at $x = 0$: $y(0) = y^{(1)}(0) = y^{(3)}(0) = 0$, $y^{(2)}(0) = 1$, $y^{(4)}(0) = -2$.	**M1**
$y = \frac{1}{2}x^2 - \frac{1}{12}x^4$ (CWO).	**A1**
	Total: 7

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Q8

9231/21, 9231/23 • October/November 2021 • Question 8

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Mark scheme.

$\cosh A = \frac{1}{2}(\mathrm{e}^{A} + \mathrm{e}^{-A})$, writes in exponential form.	**B1**
$2\cosh^2 A = \frac{1}{2}(\mathrm{e}^{2A} + 2 + \mathrm{e}^{-2A}) = \frac{1}{2}(\mathrm{e}^{2A} + \mathrm{e}^{-2A}) + 1.$	**M1**
$= \cosh 2A + 1$ (AG).	**A1**
$\frac{\mathrm{d}x}{\mathrm{d}t} = 4\sinh 2t + 3,\ \frac{\mathrm{d}y}{\mathrm{d}t} = 3\sinh 2t - 4.$	**B1**
$(4\sinh 2t + 3)^2 + (3\sinh 2t - 4)^2 = 25(\sinh^2 2t + 1) = 25\cosh^2 2t.$	**M1 A1**
$A = 2\pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (2\cosh 2t + 3t)(5\cosh 2t)\,\mathrm{d}t = \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} (20\cosh^2 2t + 30t\cosh 2t)\,\mathrm{d}t$ (AG).	**A1**
Applies $\cosh^2 A = \frac{1}{2}(\cosh 2A + 1)$: $20\int \cosh^2 2t\,\mathrm{d}t = 10\int \cosh 4t + 1\,\mathrm{d}t.$	**M1**
$10\left[\frac{1}{4}\sinh 4t + t\right]_{-\frac{1}{2}}^{\frac{1}{2}} = 10(\frac{1}{2}\sinh 2 + 1).$	**M1 A1**
Integrates by parts: $30\int_{-\frac{1}{2}}^{\frac{1}{2}} t\cosh 2t\,\mathrm{d}t = 30\left(\frac{1}{2}[t\sinh 2t]_{-\frac{1}{2}}^{\frac{1}{2}} - \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sinh 2t\,\mathrm{d}t\right).$	**M1 A1**
$15[t\sinh 2t - \frac{1}{2}\cosh 2t]_{-\frac{1}{2}}^{\frac{1}{2}} = 0.$	**A1**
$A = 10\pi\left(\frac{1}{4}(\mathrm{e}^2 - \mathrm{e}^{-2}) + 1\right)$, i.e. $64.82457\ldots$	**A1**
	Total: 14

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Q9

9231/22 • October/November 2021 • Question 8(a)(b)

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Mark scheme.

$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, $\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$.	**B1**
$1 - \left(\dfrac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = \dfrac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \dfrac{4}{(e^x + e^{-x})^2}$.	**M1**
$= \operatorname{sech}^2 x$ (AG).	**A1**
Uses substitution correctly (allow by inspection): $\int u^2 \, du = \frac{1}{3} \tanh^3 x + C$.	**M1 A1**
	Total: 5

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Q10

9231/21, 9231/22 • May/June 2022 • Question 2

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Mark scheme.

$\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$	**(a) B1**
Expands, AG: $\frac{1}{2}(e^x - e^{-x})^2 + 1 = \frac{1}{2}(e^{2x} - 2 + e^{-2x}) + 1 = \cosh 2x$	**M1 A1**
Applies identity: $2\sinh^2 x - k\sinh x + 1 = 0$	**(b) M1 A1**
Sets discriminant positive: $k^2 - 8 > 0$	**M1 A1**
$k < -\sqrt{8},\ k > \sqrt{8}$	**A1**
	Total: 8

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Q11

9231/21, 9231/23 • October/November 2022 • Question 4

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Mark scheme.

(a)

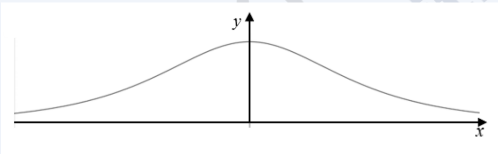
$\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.	**B1**
$\frac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}) = 1$. Expands, AG.	**M1 A1**

(b)

$\frac{d}{dx}\left(\tan^{-1}(\sinh x)\right) = \frac{\cosh x}{\sinh^2 x + 1}$, applies $\frac{d}{du}(\tan^{-1} u) = \frac{1}{u^2+1}$.	**M1 A1**
$= \frac{\cosh x}{\cosh^2 x} = \operatorname{sech} x$. AG.	**A1**

(c)

Correct shape, symmetrical about $x = 0$.	**B1**
Asymptote $y = 0$ (accept labels on sketch).	**B1**



(d)

$\sum_{r=1}^{n} \operatorname{sech} r < \int_0^n \operatorname{sech} x \, dx$. Compares sum with integral; limits correct for A1.	**M1 A1**
$= \left[\tan^{-1} \sinh x\right]_0^n = \tan^{-1} \sinh n$. AG.	**A1**

(e)

$\frac{1}{2}\pi$.	**B1**

Total: 12

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Q12

9231/21, 9231/22 • May/June 2023 • Question 7

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Mark scheme.

Applies given substitution: $\int \frac{x}{\sqrt{x^2-1}}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u$.	**M1 A1**
$\sqrt{x^2-1}+C$ (allow with "$+C$" missing; answer must be in terms of x). **(b)**	**A1**
$\cosh^{-1} r = \ln\left(r+\sqrt{r^2-1}\right)$.	**B1**
Forms sum of the areas of the rectangles: $\cosh^{-1} 2+\cosh^{-1} 3+\cdots+\cosh^{-1} N$.	**M1**
Compares with integral with correct limits: $> \int_1^N \cosh^{-1} x\,\mathrm{d}x$.	**M1**
Evaluates integral: $\int_1^N \cosh^{-1} x\,\mathrm{d}x = \left[x\cosh^{-1} x\right]_1^N - \int_1^N \frac{x}{\sqrt{x^2-1}}\,\mathrm{d}x$.	**A1**
$\sum_{r=2}^N \ln\left(r+\sqrt{r^2-1}\right) > N\ln\left(N+\sqrt{N^2-1}\right)-\sqrt{N^2-1}$ (AG). **(c)**	**A1**
Compares with integral with correct limits: $(\cosh^{-1} 1)+\cosh^{-1} 2+\cdots+\cosh^{-1}(N-1) < \int_1^N \cosh^{-1} x\,\mathrm{d}x$ (or equivalent forms).	**M1 A1**
Adds $\ln\left(N+\sqrt{N^2-1}\right)$ to both sides: $\sum_{r=2}^N \ln\left(r+\sqrt{r^2-1}\right) < (N+1)\ln\left(N+\sqrt{N^2-1}\right)-\sqrt{N^2-1}$ (or equivalent form).	**A1**
	Total: 11

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Q13

9231/21, 9231/22 • May/June 2023 • Question 8

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Mark scheme.

$\operatorname{sech} t = \dfrac{2}{e^t + e^{-t}}$, $\tanh t = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$.	**B1**
Expands, gets to $\dfrac{e^{2t} + 2 + e^{-2t} - 4}{(e^t + e^{-t})^2}$ for M1, AG: $1 - \left(\dfrac{2}{e^t + e^{-t}}\right)^2 = \dfrac{(e^t - e^{-t})^2}{(e^t + e^{-t})^2}$. **(b)**	**M1 A1**
$\dfrac{dy}{dt} = 4\tanh^3 t \operatorname{sech}^2 t$.	**B1**
Sensible attempt at derivative of x: $\dfrac{dx}{dt} = \tanh t \operatorname{sech}^2 t - \tanh t = -\tanh^3 t$.	**M1 A1**
Applies chain rule, must substitute their $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ for M1, AG: $\dfrac{dy}{dx} = \dfrac{4\tanh^3 t \operatorname{sech}^2 t}{-\tanh^3 t} = -4\operatorname{sech}^2 t$. **(c)**	**M1 A1**
Finds $\dfrac{d^2y}{dx^2}$: $\dfrac{d^2y}{dx^2} = -8\dfrac{\operatorname{sech}^2 t}{\tanh^2 t}$ (accept $-8\operatorname{cosech}^2 t$).	**M1 A1**
Sets equal to $-\frac{9}{2}$, uses identity from (a) or equivalent: $8(1 - \tanh^2 t) = \frac{9}{2}\tanh^2 t \Rightarrow \tanh^2 t = \frac{16}{25}$.	**M1 A1**
$(x, y) = \left(\frac{8}{25} + \ln\frac{3}{5}, \frac{881}{625}\right)$ $(t = \ln 3)$; A1 for each coordinate.	**A1 A1**
	Total: 14

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Q14

9231/23 • May/June 2023 • Question 5

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Mark scheme.

$\cosh x = \frac{1}{2}(e^x + e^{-x})$.	**B1**
Expands (AG): $\frac{2}{4}(e^x + e^{-x})^2 = \frac{1}{2}(e^{2x} + e^{-2x} + 2) = \cosh 2x + 1$. **(b)**	**M1 A1**
Finds integrating factor: $e^{2\int \tanh x \, dx} = e^{2 \ln \cosh x} = \cosh^2 x$.	**M1 A1**
Correct form on LHS, $\frac{d}{dx}(yI)$ for their integrating factor I, and attempt to integrate their RHS: $\frac{d}{dx}(y \cosh^2 x) = \cosh^2 x$.	**M1**
Uses $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$: $y \cosh^2 x = \int \cosh^2 x \, dx = \frac{1}{2} \int \cosh 2x + 1 \, dx = \frac{1}{2}\left(\frac{1}{2} \sinh 2x + x\right) + C$.	**M1 A1**
Finds C: $1 = C$.	**M1**
Divides through by their coefficient of y: $y = \operatorname{sech}^2 x \left(\frac{1}{4} \sinh 2x + \frac{1}{2}x + 1\right)$.	**M1 A1**
	Total: 11

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Q15

9231/21, 9231/23 • October/November 2023 • Question 3

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Mark scheme.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{1}{2}\mathrm{e}^x}{1-\frac{1}{4}\mathrm{e}^{2x}}$.	**B1**
$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\left(1-\frac{1}{4}\mathrm{e}^{2x}\right)\left(\frac{1}{2}\mathrm{e}^x\right)-\frac{1}{2}\mathrm{e}^x\left(-\frac{1}{2}\mathrm{e}^{2x}\right)}{\left(1-\frac{1}{4}\mathrm{e}^{2x}\right)^2}$.	**B1**
Evaluates derivatives at $x=0$: $f'(0)=\frac{2}{3}$, $f''(0)=\frac{10}{9}$.	**M1**
Uses logarithmic form of $\tanh^{-1}$: $f(0)=\tanh^{-1}\left(\frac{1}{2}\right)=\frac{1}{2}\ln\left(\frac{3}{2}\times 2\right)$.	**M1**
Applies $f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2$: $\frac{1}{2}\ln 3+\frac{2}{3}x+\frac{5}{9}x^2$.	**M1 A1**
	Total: 6

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Q16

9231/22 • October/November 2023 • Question 5

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Mark scheme.

(a)	
Differentiating using product rule and chain rule for M1: $\frac{dy}{dx} = -2x\,\text{sech}^2\,x\tanh x + \text{sech}^2\,x \ [= 1 - \tanh^2 x - 2x\tanh x + 2x\tanh^3 x]$.	**M1 A1**
(AG) $\text{sech}^2\,x \neq 0 \Rightarrow 2x\tanh x - 1 = 0$.	**A1**
Shows sign change (must write down values correct to at least 1dp): $2(0.7)\tanh(0.7) - 1 = -0.154 < 0$, $2(0.8)\tanh(0.8) - 1 = 0.062 > 0$.	**B1**
(b)	
Compares sum with integral (consistent limits for M1): $\sum_{r=2}^{n} r\,\text{sech}^2\,r < \int_1^n x\,\text{sech}^2\,x\,dx$.	**M1 A1**
Integrates by parts: $\int_1^n x\,\text{sech}^2\,x\,dx = [x\tanh x]_1^n - \int_1^n \tanh x\,dx$.	**M1 A1**
$= [x\tanh x + \ln\text{sech}\,x]_1^n$.	**A1**
(AG, must have gained all previous marks) $= n\tanh n + \ln\text{sech}\,n - (\tanh 1 + \ln\text{sech}\,1)$.	**A1**
	Total: 10

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Q17

9231/22 • October/November 2023 • Question 7

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Mark scheme.

(a)

$\cosh 2A = \frac{1}{2}\left(e^{2A} + e^{-2A}\right)$, $\sinh A = \frac{1}{2}\left(e^{A} - e^{-A}\right)$.	**B1**
Expands, AG (A0 for mixing variables e.g. $\sinh A = \frac{1}{2}(e^{x} - e^{-x})$): $2\sinh^2 A = \frac{1}{2}\left(e^{A} - e^{-A}\right)^2 = \frac{1}{2}\left(e^{2A} - 2 + e^{-2A}\right) = \cosh 2A - 1$.	**M1 A1**

(b)

Correct formula with correct limits (correct limits for M1, limits may be recovered): $S = \int_0^{\frac{2}{3}} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^{\frac{2}{3}} x^2\sqrt{1 + 4x^2}\, dx$.	**M1 A1**
Applies given substitution to their expression with correct limits (limits may be recovered): $S = \pi \int_0^{\sinh^{-1}\frac{4}{3}} \frac{1}{4}\sinh^2 u\sqrt{1 + \sinh^2 u}\cosh u\, du$.	**M1**
Must be simplified: $= \frac{1}{4}\pi \int_0^{\sinh^{-1}\frac{4}{3}} \sinh^2 u \cosh^2 u\, du$.	**A1**
Applies $\sinh 2u = 2\sinh u \cosh u$ (may use double angle formulae for cosh and sinh instead): $= \frac{1}{16}\pi \int_0^{\sinh^{-1}\frac{4}{3}} \sinh^2 2u\, du$.	**M1**
Applies $\sinh^2 A = \frac{1}{2}(\cosh 2A - 1)$; S must have the form $a\int \sinh^2 u\cosh^2 u\, du$: $= \frac{1}{32}\pi \int_0^{\sinh^{-1}\frac{4}{3}} \cosh 4u - 1\, du$.	**M1**
$= \frac{1}{32}\pi \left[\frac{1}{4}\sinh 4u - u\right]_0^{\sinh^{-1}\frac{4}{3}}$.	**A1**
$\sinh^{-1}\frac{4}{3} = \ln 3$.	**B1**
(AG) $S = \frac{1}{32}\pi\left(\frac{1}{8}\left(e^{\ln 81} - e^{-\ln 81}\right) - \ln 3\right) = \frac{1}{32}\pi\left(\frac{1}{8}\left(81 - \frac{1}{81}\right) - \ln 3\right) = \frac{1}{32}\pi\left(\frac{820}{81} - \ln 3\right)$.	**A1**

Total: 12

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Q18

9231/21, 9231/22 • May/June 2025 • Question 6

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Mark scheme.

(a)	
Writes in exponential form: $\tanh u = \dfrac{e^u - e^{-u}}{e^u + e^{-u}}$, $\operatorname{sech} u = \dfrac{2}{e^u + e^{-u}}$.	**B1**
Writes over common denominator: $1 - \left(\dfrac{e^u - e^{-u}}{e^u + e^{-u}}\right)^2 = \dfrac{(e^u + e^{-u})^2 - (e^u - e^{-u})^2}{(e^u + e^{-u})^2} = \dfrac{4}{(e^u + e^{-u})^2}$.	**M1**
$= \operatorname{sech}^2 u$. Withhold this mark if working with LHS and RHS simultaneously.	**A1**
(b)	
Differentiates implicitly: $u = \operatorname{sech}^{-1} t \Rightarrow \operatorname{sech} u = t \Rightarrow -\tanh u \operatorname{sech} u \dfrac{du}{dt} = 1$.	**M1 A1**
Uses identity from (a): $\dfrac{du}{dt} = -\dfrac{1}{\tanh u \operatorname{sech} u} = -\dfrac{1}{t\sqrt{1-t^2}}$, AG.	**M1 A1**
(c)	
$\dfrac{dy}{dt} = \operatorname{sech}^{-1} t - \dfrac{1}{\sqrt{1-t^2}}$, $\dfrac{dx}{dt} = \dfrac{1}{1-t^2}$.	**B1 B1**
Uses chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = -\sqrt{1-t^2} + (1-t^2)\operatorname{sech}^{-1} t$, AG.	**M1 A1**
(d)	
Applies product rule: $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{t}{\sqrt{1-t^2}} - \dfrac{1-t^2}{t\sqrt{1-t^2}} - 2t \operatorname{sech}^{-1} t$.	**M1 A1**
Uses chain rule: $\dfrac{d^2y}{dx^2} = t\sqrt{1-t^2} - \dfrac{(1-t^2)^{3/2}}{t} - 2t(1-t^2)\operatorname{sech}^{-1} t$.	**M1 A1**
	Total: 15

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Q19

9231/23 • May/June 2025 • Question 2

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Mark scheme.

(a)

$\tanh t = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$, $\operatorname{sech} t = \dfrac{2}{e^t + e^{-t}}$.	**B1**
Writes in a single fraction over common denominator: $\left(\dfrac{e^t - e^{-t}}{e^t + e^{-t}}\right)^2 + \dfrac{4}{(e^t + e^{-t})^2} = \dfrac{e^{2t} + e^{-2t} + 2}{(e^t + e^{-t})^2} = 1$, AG.	**M1**
Withhold A1 mark if they used mixed variables within a single line of working.	**A1**

(b)

$\dfrac{dx}{dt} = \tanh t$.	**B1**
$\dfrac{dy}{dt} = \dfrac{\cosh t}{1 + \sinh^2 t} = \operatorname{sech} t$. Applies $1 + \sinh^2 t = \cosh^2 t$.	**M1 A1**
Substitutes their derivatives into a correct formula for arc length, e.g. $\int \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$: $\int_0^1 \sqrt{\tanh^2 t + \operatorname{sech}^2 t}\, dt = \int_0^1 1\, dt$.	**M1**
$= 1$.	**A1**
	Total: 8

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Q20

9231/24 • May/June 2025 • Question 8

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Mark scheme.

(a)	
Correct shape and position (ignore $x < 0$ on their sketch): curve rising from origin, concave down, approaching $y = 1$.	**B1**
States the equation of the asymptote: $y = 1$. Ignore $y = -1$.	**B1**
(b)	
Compares sum with integral. Limits on integral must be present and correct for M1: $\sum_{r=1}^{N} \tanh r > \int_0^N \tanh x \, \mathrm{d}x$.	**M1 A1**
Evaluates integral, AG. Must see $\ln(\cosh x)$: $\int_0^N \tanh x \, \mathrm{d}x = [\ln(\cosh x)]_0^N = \ln(\cosh N)$.	**A1**
(c)(i)	
Correct formula for surface area (missing limits is M1 A0; y outside of integral is M0): $S = 2\pi \int_0^{\frac{1}{2}\ln 3} y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x = 2\pi \int_0^{\frac{1}{2}\ln 3} \tanh(x)\sqrt{1 + \operatorname{sech}^4(x)} \, \mathrm{d}x$.	**M1 A1**
Finds $\frac{\mathrm{d}u}{\mathrm{d}x}$ (sign error gets M1 A0): $u = \sqrt{1 + \operatorname{sech}^4 x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -2(1 + \operatorname{sech}^4 x)^{-1/2} \operatorname{sech}^4 x \tanh x$.	**M1 A1**
Finds u when $x = \frac{1}{2}\ln 3$ (need to see working to justify given limit): $x = \frac{1}{2}\ln 3 \Rightarrow \cosh x = \frac{2}{3}\sqrt{3}$, $\operatorname{sech} x = \frac{1}{2}\sqrt{3} \Rightarrow u = \sqrt{1 + \frac{9}{16}}$.	**M1 A1**
AG, must have gained all previous marks: $S = -\pi \int_{\sqrt{2}}^{5/4} u \frac{u}{u^2 - 1} \, \mathrm{d}u = \pi \int_{5/4}^{\sqrt{2}} \frac{u^2}{u^2 - 1} \, \mathrm{d}u$.	**A1**
(c)(ii)	
Uses partial fractions or $\frac{u^2}{u^2 - 1} = 1 + \frac{1}{u^2 - 1}$: $= \frac{1}{2}\pi \int_{5/4}^{\sqrt{2}} 2 + \frac{1}{u - 1} - \frac{1}{u + 1} \, \mathrm{d}u = \pi\left[u + \frac{1}{2}\ln(u - 1) - \frac{1}{2}\ln(u + 1)\right]_{5/4}^{\sqrt{2}}$.	**M1 A1**
OE: $= \pi\left(\sqrt{2} + \frac{1}{2}\ln\left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right) - \frac{5}{4} - \frac{1}{2}\ln\left(\frac{1}{9}\right)\right) = \pi\left(\sqrt{2} - \frac{5}{4} + \ln 3 + \frac{1}{2}\ln(3 - 2\sqrt{2})\right)$.	**A1**
	Total: 15

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Q21

9231/21, 9231/23 • October/November 2025 • Question 2

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Mark scheme.

Applies chain rule: $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{1+\sinh t}{\cosh t}$.	**(a) M1**
Correct result.	**A1**
Finds $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{\cosh t(\cosh t) - (1+\sinh t)\sinh t}{\cosh^2 t} = \dfrac{1-\sinh t}{\cosh^2 t}$.	**(b) M1**
Correct intermediate result.	**A1**
Applies chain rule: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \times \dfrac{dt}{dx} = \dfrac{1-\sinh t}{\cosh^3 t}$.	**M1**
AG, shown.	**A1**
Finds Maclaurin's series: $y = y(0) + xy'(0) + \dfrac{x^2}{2!}y''(0)$. M1 for $y(0), y'(0), y''(0)$ evaluated.	**(c) M1**
$y = 1 + x + \frac{1}{2}x^2$. A0 if 2! not simplified to 2.	**A1**
	Total: 8

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Q22

9231/21, 9231/23 • October/November 2025 • Question 7(a)

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Mark scheme.

Applies $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$: $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{2}\left(\frac{1-x}{1+x}\right)\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}$.	**M1**
Correct differentiation.	**A1**
$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)\frac{2}{(1-x)^2} = \frac{1}{1-x^2}$. AG.	**A1**
$\tanh y = x \Rightarrow \operatorname{sech}^2 y\,\frac{dy}{dx} = 1$ (M1 A1); $\frac{dy}{dx} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}$ (A1). AG.	**Alt**
	Total: 3

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Q23

9231/24 • October/November 2025 • Question 2

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Mark scheme.

Sets equal and uses $\cosh 2x = 1+2\sinh^2 x$: $1+2\sinh^2 x = 6\sinh^2 x \Rightarrow \sinh x = \pm\frac{1}{2}$. Accept $\cosh x = \frac{1}{2}\sqrt{5}$ or $\tanh x = \pm\frac{1}{\sqrt{5}}$, or exponential form leading to $e^{4x} - 3e^{2x} + 1 = 0$.	**(a) M1** **A1**
Uses logarithmic form: $x = \sinh^{-1}(\pm\frac{1}{2}) = \ln(\pm\frac{1}{2}+\sqrt{\frac{1}{4}+1})$, or $e^{2x} = \frac{1}{2}(3\pm\sqrt{5})$.	**M1**
$x = \ln(\pm\frac{1}{2} + \frac{1}{2}\sqrt{5})$. Must be exact and in the form $\ln a$ simplified.	**A1**
C_1 or C_2 correct.	**(b) B1**
C_2 and intersections with C_1 correct.	**B1**
	Total: 6

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