



Further Math with Ali Hashir

Differentiation

Topical Questions (CIE 9231)

Compiled from CIE 9231 Further Mathematics past papers (2015–2025), sorted by topic. Each question is numbered for this booklet, with its original paper code and session shown beneath.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at vectorspacepk@gmail.com.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Q1

9231/11, 9231/12 • May/June 2015 • Question 6

A curve has equation $x^2 - 6xy + 25y^2 = 16$. Show that $\frac{dy}{dx} = 0$ at the point $(3, 1)$. [4]

By finding the value of $\frac{d^2y}{dx^2}$ at the point $(3, 1)$, determine the nature of this turning point. [5]

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Q2

9231/13 • May/June 2015 • Question 7

The curve C has equation $x^2 + 2xy - 4y^2 + 20 = 0$. Show that if the tangent to C at the point (x, y) is parallel to the x -axis then $x + y = 0$. [3]

Hence find the coordinates of the stationary points on C , and determine their nature. [7]

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Q3

9231/11, 9231/12, 9231/13 • October/November 2015 • Question 1

The curve C is defined parametrically by

$$x = 2 \cos^3 t \quad \text{and} \quad y = 2 \sin^3 t, \quad \text{for } 0 < t < \frac{1}{2}\pi.$$

Show that, at the point with parameter t ,

$$\frac{d^2y}{dx^2} = \frac{1}{6} \sec^4 t \operatorname{cosec} t.$$

[4]

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Q4

9231/11, 9231/12, 9231/13 • October/November 2016 • Question 8

A curve C has equation $x^2 + 4xy - y^2 + 20 = 0$. Show that, at stationary points on C , $x = -2y$. [3]

Find the coordinates of the stationary points on C , and determine their nature by considering the value of $\frac{d^2y}{dx^2}$ at the stationary points. [8]

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Q5

9231/11, 9231/12 • May/June 2017 • Question 3

A curve C has equation $\tan y = x$, for $x > 0$.

(i) Use implicit differentiation to show that

$$\frac{d^2y}{dx^2} = -2x \left(\frac{dy}{dx} \right)^2.$$

[3]

(ii) Hence find the value of $\frac{d^2y}{dx^2}$ at the point $\left(1, \frac{1}{4}\pi\right)$ on C .

[2]

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Q6

9231/13 • May/June 2017 • Question 4

A curve C has equation $x^3 - 3xy + y^2 = 4$. Find the value of $\frac{d^2y}{dx^2}$ at the point $(0, 2)$ of C .
[7]

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Q7

9231/11, 9231/12, 9231/13 • October/November 2017 • Question 5

The curve C has equation $2x^3 + 3x^2y - 3y^3 - 16 = 0$.

- (i) Find the coordinates of the point A on C at which $\frac{dy}{dx} = 0$ and $x \neq 0$. [5]
- (ii) Find the value of $\frac{d^2y}{dx^2}$ at A . [3]

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Q8

9231/13 • May/June 2018 • Question 1

The variables x and y are such that $y = -1$ when $x = 0$ and

$$\left(x + \frac{dy}{dx}\right)^3 = y^2 + x.$$

(i) Find the value of $\frac{dy}{dx}$ when $x = 0$. [1]

(ii) Find also the value of $\frac{d^2y}{dx^2}$ when $x = 0$. [4]

Q9

9231/12 • October/November 2018 • Question 11

The curve C is defined parametrically by

$$x = 18t - t^2 \quad \text{and} \quad y = 8t^{\frac{3}{2}},$$

where $0 < t \leq 4$.

Show that at all points of C ,

$$\frac{d^2y}{dx^2} = \frac{3(9+t)}{2t^{\frac{1}{2}}(9-t)^3}.$$

[4]

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Q10

9231/11, 9231/12 • May/June 2019 • Question 1

A curve C has equation $\cos y = x$, for $-\pi < x < \pi$.

(i) Use implicit differentiation to show that

$$\frac{d^2y}{dx^2} = -\cot y \left(\frac{dy}{dx} \right)^2.$$

[4]

(ii) Hence find the exact value of $\frac{d^2y}{dx^2}$ at the point $\left(\frac{1}{2}, \frac{1}{3}\pi\right)$ on C .

[2]

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Q11

9231/21, 9231/22 • May/June 2020 • Question 2

It is given that $y = 2^x$.

- (a) By differentiating $\ln y$ with respect to x , show that $\frac{dy}{dx} = 2^x \ln 2$. [3]
- (b) Write down $\frac{d^2y}{dx^2}$. [1]
- (c) Hence find the first three terms in the Maclaurin's series for 2^x . [3]

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Q12

9231/21, 9231/23 • October/November 2020 • Question 1

(a) By differentiating e^{-x^2} , find the Maclaurin's series for e^{-x^2} up to and including the term in x^2 . [5]

(b) Deduce an approximation to $\int_0^{\frac{1}{5}} e^{-x^2} dx$, giving your answer as a rational fraction in its lowest terms. [2]

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Q13

9231/21, 9231/23 • October/November 2020 • Question 5

It is given that

$$x = \sinh^{-1} t, \quad y = \cos^{-1} t,$$

where $-1 < t < 1$.

(a) By differentiating $\cos y$ with respect to t , show that $\frac{dy}{dt} = -\frac{1}{\sqrt{1-t^2}}$. [4]

(b) Find $\frac{d^2y}{dx^2}$ in terms of t , simplifying your answer. [5]

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Q14

9231/22 • October/November 2020 • Question 1

Find the Maclaurin's series for $\tan\left(x + \frac{1}{4}\pi\right)$ up to and including the term in x^2 . [5]

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Q15

9231/22 • October/November 2020 • Question 5

The curve C has equation

$$y^2 + (xy + 1)^2 = 5.$$

(a) Show that, at the point $(1, 1)$ on C , $\frac{dy}{dx} = -\frac{2}{3}$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(1, 1)$. [5]

Q16

9231/21, 9231/23 • October/November 2021 • Question 1

Find the Maclaurin's series for $e^x \tan x$ from first principles up to and including the term in x^2 . [5]

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Q17

9231/21, 9231/23 • October/November 2021 • Question 3

The curve C has equation

$$xy^3 - 4x^3y = 3.$$

(a) Show that, at the point $(-1, 1)$ on C , $\frac{dy}{dx} = 11$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(-1, 1)$. [5]

Q18

9231/22 • October/November 2021 • Question 1

It is given that $y = \sinh(x^2) + \cosh(x^2)$.

(a) Use standard results from the list of formulae (MF19) to find the Maclaurin's series for y in terms of x up to and including the term in x^4 . [2]

(b) Deduce the value of $\frac{d^4y}{dx^4}$ when $x = 0$. [1]

(c) Use your answer to part (a) to find an approximation to $\int_0^{\frac{1}{2}} y \, dx$, giving your answer as a rational fraction in its lowest terms. [2]

Q19

9231/22 • October/November 2021 • Question 5(b)

The curve C has parametric equations

$$x = 3t + 2t^{-1} + at^3, \quad y = 4t - \frac{3}{2}t^{-1} + bt^3, \quad \text{for } 1 \leq t \leq 2,$$

where a and b are constants.

(b) It is given instead that $a = b = 0$.

Find the value of $\frac{d^2y}{dx^2}$ when $t = 1$. [4]

Q20

9231/21, 9231/22 • May/June 2022 • Question 5

The variables x and y are such that $y = 0$ when $x = 0$ and

$$(x + 1)y + (x + y + 1)^3 = 1.$$

(a) Show that $\frac{dy}{dx} = -\frac{3}{4}$ when $x = 0$. [3]

(b) Find the Maclaurin's series for y up to and including the term in x^2 . [7]

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Q21

9231/23 • May/June 2022 • Question 2(a)

Find the coefficient of x^2 in the Maclaurin's series for $-\ln \cos x$.

[4]

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Q22

9231/23 • May/June 2022 • Question 4

It is given that

$$x = -t + \tan^{-1} t \quad \text{and} \quad y = t + \sinh^{-1} t.$$

(a) Show that $\frac{dy}{dx} = -\frac{t^2 + 1 + \sqrt{t^2 + 1}}{t^2}$. [4]

(b) Find the value of $\frac{d^2y}{dx^2}$ when $t = \frac{3}{4}$. [5]

Q23

9231/21, 9231/23 • October/November 2022 • Question 1

Find the Maclaurin's series for $\ln(1 + e^x)$ up to and including the term in x^2 . [5]

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Q24

9231/22 • October/November 2022 • Question 2

A curve has equation

$$(x + 1)y + y^2 = 2.$$

(a) Show that $\frac{dy}{dx} = -\frac{2}{3}$ at the point $(0, -2)$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(0, -2)$. [4]

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Q25

9231/22 • October/November 2022 • Question 3(b)

A curve has equation $y = e^x + \frac{1}{4}e^{-x}$, for $0 \leq x \leq 1$.

(b) Using standard results from the list of formulae (MF19), or otherwise, find the Maclaurin's series for $e^x + \frac{1}{4}e^{-x}$ up to and including the term in x^2 . [2]

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Q26

9231/23 • May/June 2023 • Question 1

(a) Find the Maclaurin series for $\sin^{-1} x$ up to and including the term in x^3 . [5]

(b) Deduce an approximation to $\int_0^{\frac{1}{5}} \frac{1}{\sqrt{1-u^2}} du$, giving your answer as a fraction. [1]

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Q27

9231/23 • May/June 2023 • Question 4

The curve C has equation

$$4y^3 + (x + y)^6 = 109.$$

(a) Show that, at the point $(-4, 3)$ on C , $\frac{dy}{dx} = \frac{1}{17}$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(-4, 3)$. [5]

Q28

9231/21, 9231/23 • October/November 2023 • Question 5(b)

The curve C has parametric equations

$$x = \frac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}}, \quad y = 2t + 5, \quad \text{for } 0 < t \leq 3.$$

(b) Find the set of values of t for which $\frac{d^2y}{dx^2} > 0$. [5]

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Q29

9231/22 • October/November 2023 • Question 1

Find the Maclaurin's series for $\ln(x+2) + \ln(x^2+5)$ up to and including the term in x^2 . [5]

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Q30

9231/22 • October/November 2023 • Question 2

It is given that

$$x = 1 + \frac{1}{t} \quad \text{and} \quad y = te^t.$$

(a) Show that $\frac{dy}{dx} = -e^t(t^3 + t^2)$. [3]

(b) Find $\frac{d^2y}{dx^2}$ in terms of t . [4]

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Q31

9231/21, 9231/22 • May/June 2024 • Question 2

Find the Maclaurin's series for $e^{1+x^2} + e^{1-x}$ up to and including the term in x^2 . [4]

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Q32

9231/21, 9231/22 • May/June 2024 • Question 3

It is given that

$$x = \sin^{-1} t \quad \text{and} \quad y = t \cos^{-1} t, \quad \text{for } 0 \leq t < 1.$$

(a) Show that $\frac{dy}{dx} = -t + \sqrt{1-t^2} \cos^{-1} t.$ [3]

(b) Find $\frac{d^2y}{dx^2}$ in terms of $t.$ [4]

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Q33

9231/23 • May/June 2024 • Question 3

The curve C has equation

$$x^3 + 2xy + 8y^3 = -12.$$

(a) Show that, at the point $(-2, -1)$ on C , $\frac{dy}{dx} = -\frac{1}{2}$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(-2, -1)$. [5]

Q34

9231/21, 9231/23 • October/November 2024 • Question 2

It is given that

$$x = 1 + \frac{1}{t} \quad \text{and} \quad y = \cos^{-1} t \quad \text{for } 0 < t < 1.$$

(a) Show that $\frac{dy}{dx} = \frac{t^2}{\sqrt{1-t^2}}$. [2]

(b) Show that $\frac{d^2y}{dx^2} = -t^a(1-t^2)^b(2-t^2)$, where a and b are constants to be determined. [4]

Q35

9231/22 • October/November 2024 • Question 2

The curve C has equation

$$4y^2 + 4\ln(xy) = 1.$$

(a) Show that, at the point $\left(2, \frac{1}{2}\right)$ on C , $\frac{dy}{dx} = -\frac{1}{6}$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $\left(2, \frac{1}{2}\right)$. [4]

Q36

9231/23 • May/June 2025 • Question 1

Find the Maclaurin's series for $e^{\left(\frac{1}{x+2}\right)}$ up to and including the term in x^2 . [5]

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Q37

9231/23 • May/June 2025 • Question 3

The curve C has equation

$$9y^2 - 3 \sinh^{-1}(xy) = 1 - 3 \ln 3.$$

(a) Show that, at the point $\left(4, \frac{1}{3}\right)$ on C , $\frac{dy}{dx} = -\frac{1}{2}$. [4]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $\left(4, \frac{1}{3}\right)$. [5]

Q38

9231/24 • May/June 2025 • Question 4

A curve has parametric equations

$$x = t^3 - t^2 + t - 1 \quad \text{and} \quad y = te^t.$$

(a) Show that 1 is the only real value of t for which $x = 0$. [1]

(b) Show that $\frac{dy}{dx} = \frac{(t+1)e^t}{3t^2 - 2t + 1}$. [3]

(c) Find the Maclaurin's series for y up to and including the term in x^2 . [6]

Q39

9231/22 • October/November 2025 • Question 3

The variables x and y are such that $y = 2$ when $x = -1$ and

$$x^2y + (x + y)^3 = 3.$$

(a) Show that $\frac{dy}{dx} = \frac{1}{4}$ when $x = -1$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ when $x = -1$. [6]

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Q40

9231/22 • October/November 2025 • Question 7(a)

Show that $\frac{d}{dx} \left(\frac{1}{2}x\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{1}{2}x\right) \right) = \sqrt{4-x^2}$. [3]

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Q41

9231/24 • October/November 2025 • Question 7(b)

Show that

$$\frac{d^2y}{dx^2} = \frac{a \cot t \operatorname{cosec} t (\operatorname{cosec}^2 t + 1)}{(b \operatorname{cosec} t + c \cot^2 t)^n},$$

where a , b , c and n are integers to be determined. (Using the curve C from part (a) with parametric equations $x = 8 \ln(\tan \frac{1}{2}t) - 3 \cot t - 3t$, $y = 6 \ln(\tan \frac{1}{2}t) + 4 \cot t + 4t$.) [5]

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Answers available in the companion mark scheme booklet.

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