



Further Math with Ali Hashir

Rational Functions

Topical Questions (CIE 9231)

Compiled from CIE 9231 Further Mathematics Paper 1 past papers (2015–2025), sorted by topic. Each question is numbered for this booklet, with its original paper code and session shown beneath.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at vectorspacepk@gmail.com.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Q1

9231/13 • May/June 2015 • Question 10

The curve C has equation $y = \frac{4x^2 - 3x}{x^2 + 1}$. Verify that the equation of C may be written in the form

$$y = -\frac{1}{2} + \frac{(3x - 1)^2}{2(x^2 + 1)} \quad \text{and also in the form} \quad y = \frac{9}{2} - \frac{(x + 3)^2}{2(x^2 + 1)}.$$

[3]

Hence show that $-\frac{1}{2} \leq y \leq \frac{9}{2}$. [2]

Without differentiating, write down the coordinates of the turning points of C . [2]

State the equation of the asymptote of C . [1]

Sketch the graph of C , stating the coordinates of the intersections with the coordinate axes and the asymptote. [3]

Q2

9231/11, 9231/12, 9231/13 • October/November 2015 • Question 8

The curve C has equation $y = \frac{2x^2 + kx}{x + 1}$, where k is a constant. Find the set of values of k for which C has no stationary points. [5]

For the case $k = 4$, find the equations of the asymptotes of C and sketch C , indicating the coordinates of the points where C intersects the coordinate axes. [6]

Q3

9231/11, 9231/12 • May/June 2016 • Question 7

A curve C has equation $y = \frac{x^2}{x-2}$. Find the equations of the asymptotes of C . [3]

Show that there are no points on C for which $0 < y < 8$. [4]

Sketch C , giving the coordinates of the turning points. [3]

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Q4

9231/13 • May/June 2016 • Question 5

The curve C has equation $y = \frac{x+2}{x^2-9}$. Show that $\frac{dy}{dx} < 0$ at all points on C . [3]

State the equations of the asymptotes of C . [2]

Sketch C , showing the coordinates of any points of intersection with the coordinate axes. [3]

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Q5

9231/11, 9231/12 • May/June 2017 • Question 9

The curve C has equation $y = \frac{x^2 - 3x + 6}{1 - x}$.

- (i) Find the equations of the asymptotes of C . [3]
- (ii) Find the coordinates of the turning points of C . [3]
- (iii) Find the coordinates of any intersections with the coordinate axes. [2]
- (iv) Sketch C . [3]

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Q6

9231/11, 9231/12, 9231/13 • October/November 2017 • Question 9

The curve C has equation

$$y = \frac{3x - 9}{(x - 2)(x + 1)}.$$

- (i) Find the equations of the asymptotes of C . [2]
- (ii) Show that there is no point on C for which $\frac{1}{3} < y < 3$. [4]
- (iii) Find the coordinates of the turning points of C . [3]
- (iv) Sketch C . [3]

Q7

9231/11, 9231/12 • May/June 2018 • Question 6

The curve C has equation

$$y = \frac{x^2 + b}{x + b},$$

where b is a positive constant.

- (i) Find the equations of the asymptotes of C . [3]
- (ii) Show that C does not intersect the x -axis. [1]
- (iii) Justifying your answer, find the number of stationary points on C . [2]
- (iv) Sketch C . Your sketch should indicate the coordinates of any points of intersection with the y -axis. You do not need to find the coordinates of any stationary points. [3]

Q8

9231/13 • May/June 2018 • Question 4

The curve C has equation

$$y = \frac{x^2 + 7x + 6}{x - 2}.$$

- (i) Find the coordinates of the points of intersection of C with the axes. [2]
- (ii) Find the equation of each of the asymptotes of C . [3]
- (iii) Sketch C . [3]

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Q9

9231/11, 9231/13 • October/November 2018 • Question 6

The curve C has equation

$$y = \frac{x^2 + ax - 1}{x + 1},$$

where a is constant and $a > 1$.

- (i) Find the equations of the asymptotes of C . [3]
- (ii) Show that C intersects the x -axis twice. [1]
- (iii) Justifying your answer, find the number of stationary points on C . [2]
- (iv) Sketch C , stating the coordinates of its point of intersection with the y -axis. [3]

Q10

9231/12 • October/November 2018 • Question 9

The curve C has equation

$$y = \frac{5x^2 + 5x + 1}{x^2 + x + 1}.$$

- (i) Find the equation of the asymptote of C . [2]
- (ii) Show that, for all real values of x , $-\frac{1}{3} \leq y < 5$. [4]
- (iii) Find the coordinates of any stationary points of C . [2]
- (iv) Sketch C , stating the coordinates of any intersections with the y -axis. [2]

Q11

9231/11, 9231/12 • May/June 2019 • Question 10

The curves C_1 and C_2 have equations

$$y = \frac{ax}{x+5} \quad \text{and} \quad y = \frac{x^2 + (a+10)x + 5a + 26}{x+5}$$

respectively, where a is a constant and $a > 2$.

- (i) Find the equations of the asymptotes of C_1 . [2]
- (ii) Find the equation of the oblique asymptote of C_2 . [2]
- (iii) Show that C_1 and C_2 do not intersect. [2]
- (iv) Find the coordinates of the stationary points of C_2 . [3]
- (v) Sketch C_1 and C_2 on a single diagram. [You do not need to calculate the coordinates of any points where C_2 crosses the axes.] [3]

Q12

9231/13 • May/June 2019 • Question 6

The curve C has equation

$$y = \frac{x^2}{kx - 1},$$

where k is a positive constant.

- (i) Obtain the equations of the asymptotes of C . [3]
- (ii) Find the coordinates of the stationary points of C . [3]
- (iii) Sketch C . [3]

Q13

9231/11, 9231/12, 9231/13 • October/November 2019 • Question 4

The line $y = 2x + 1$ is an asymptote of the curve C with equation

$$y = \frac{x^2 + 1}{ax + b}.$$

- (i) Find the values of the constants a and b . [3]
- (ii) State the equation of the other asymptote of C . [1]
- (iii) Sketch C . [Your sketch should indicate the coordinates of any points of intersection with the y -axis. You do not need to find the coordinates of any stationary points.] [3]

Q14

9231/11, 9231/12 • May/June 2020 • Question 1

Let a be a positive constant.

(a) Sketch the curve with equation $y = \frac{ax}{x+7}$. [2]

(b) Sketch the curve with equation $y = \left| \frac{ax}{x+7} \right|$ and find the set of values of x for which $\left| \frac{ax}{x+7} \right| > \frac{a}{2}$. [4]

Q15

9231/11, 9231/12 • May/June 2020 • Question 3

The curve C has equation $y = \frac{x^2}{2x + 1}$.

- (a) Find the equations of the asymptotes of C . [3]
- (b) Find the coordinates of the stationary points on C . [3]
- (c) Sketch C . [3]

Q16

9231/13 • May/June 2020 • Question 6

The curve C has equation $y = \frac{10 + x - 2x^2}{2x - 3}$.

(a) Find the equations of the asymptotes of C . [3]

(b) Show that C has no turning points. [3]

(c) Sketch C , stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left| \frac{10 + x - 2x^2}{2x - 3} \right|$ and find in exact form the set of values of x for which $\left| \frac{10 + x - 2x^2}{2x - 3} \right| < 4$. [6]

Q17

9231/11, 9231/13 • October/November 2020 • Question 6

The curve C has equation $y = \frac{x^2 + x - 1}{x - 1}$.

- (a) Find the equations of the asymptotes of C . [3]
- (b) Show that there is no point on C for which $1 < y < 5$. [4]
- (c) Find the coordinates of the intersections of C with the axes, and sketch C . [3]
- (d) Sketch the curve with equation $y = \left| \frac{x^2 + x - 1}{x - 1} \right|$. [2]

Q18

9231/12 • October/November 2020 • Question 6

Let a be a positive constant.

- (a) The curve C_1 has equation $y = \frac{x-a}{x-2a}$. [2]

Sketch C_1 .

The curve C_2 has equation $y = \left(\frac{x-a}{x-2a}\right)^2$. The curve C_3 has equation $y = \left|\frac{x-a}{x-2a}\right|$.

- (b) (i) Find the coordinates of any stationary points of C_2 . [3]

(ii) Find also the coordinates of any points of intersection of C_2 and C_3 . [3]

- (c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which $\left(\frac{x-a}{x-2a}\right)^2 \leq \left|\frac{x-a}{x-2a}\right|$. [5]

Q19

9231/11, 9231/12 • May/June 2021 • Question 7

The curve C has equation $y = \frac{x^2 + x + 9}{x + 1}$.

- (a) Find the equations of the asymptotes of C . [3]
- (b) Find the coordinates of the stationary points on C . [4]
- (c) Sketch C , stating the coordinates of any intersections with the axes. [3]
- (d) Sketch the curve with equation $y = \left| \frac{x^2 + x + 9}{x + 1} \right|$ and find the set of values of x for which $2|x^2 + x + 9| > 13|x + 1|$. [5]

Q20

9231/13 • May/June 2021 • Question 7

The curve C has equation $y = \frac{x^2 - x - 3}{1 + x - x^2}$.

- (a) Find the equations of the asymptotes of C . [2]
- (b) Find the coordinates of any stationary points on C . [3]
- (c) Sketch C , stating the coordinates of the intersections with the axes. [3]
- (d) Sketch the curve with equation $y = \left| \frac{x^2 - x - 3}{1 + x - x^2} \right|$ and find in exact form the set of values of x for which $\left| \frac{x^2 - x - 3}{1 + x - x^2} \right| < 3$. [6]

Q21

9231/11, 9231/13 • October/November 2021 • Question 7

The curve C has equation $y = \frac{4x + 5}{4 - 4x^2}$.

- (a) Find the equations of the asymptotes of C . [2]
- (b) Find the coordinates of any stationary points on C . [4]
- (c) Sketch C , stating the coordinates of the intersections with the axes. [3]
- (d) Sketch the curve with equation $y = \left| \frac{4x + 5}{4 - 4x^2} \right|$ and find in exact form the set of values of x for which $4|4x + 5| > 5|4 - 4x^2|$. [6]

Q22

9231/12 • October/November 2021 • Question 6

The curve C has equation $y = \frac{x^2}{x-3}$.

- (a) Find the equations of the asymptotes of C . [3]
- (b) Show that there is no point on C for which $0 < y < 12$. [4]
- (c) Sketch C . [2]
- (d) (i) Sketch the graphs of $y = \left| \frac{x^2}{x-3} \right|$ and $y = |x| - 3$ on a single diagram, stating the coordinates of the intersections with the axes. [4]
- (ii) Use your sketch to find the set of values of c for which $\left| \frac{x^2}{x-3} \right| \leq |x| + c$ has no solution. [1]

Q23

9231/11, 9231/12 • May/June 2022 • Question 5

The curve C has equation $y = \frac{2x^2 - x - 1}{x^2 + x + 1}$.

- (a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote of C . [3]
- (b) Find the coordinates of the stationary points on C . [4]
- (c) Sketch C , stating the coordinates of the intersections with the axes. [3]
- (d) Sketch the curve with equation $y = \left| \frac{2x^2 - x - 1}{x^2 + x + 1} \right|$ and state the set of values of k for which $\left| \frac{2x^2 - x - 1}{x^2 + x + 1} \right| = k$ has 4 distinct real solutions. [2]

Q24

9231/13 • May/June 2022 • Question 1

(a) Sketch the curve with equation $y = \frac{x+1}{x-1}$. [2]

(b) Sketch the curve with equation $y = \frac{|x|+1}{|x|-1}$ and find the set of values of x for which $\frac{|x|+1}{|x|-1} < -2$. [4]

Q25

9231/13 • May/June 2022 • Question 3

A curve C has equation $y = \frac{ax^2 + x - 1}{x - 1}$, where a is a positive constant.

- (a) Find the equations of the asymptotes of C . [3]
- (b) Show that there is no point on C for which $1 < y < 1 + 4a$. [4]
- (c) Sketch C . You do not need to find the coordinates of the intersections with the axes. [3]

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Q26

9231/11, 9231/13 • October/November 2022 • Question 7

The curve C has equation $y = \frac{5x^2}{5x - 2}$.

(a) Find the equations of the asymptotes of C . [3]

(b) Find the coordinates of the stationary points on C . [4]

(c) Sketch C . [3]

(d) Sketch the curve with equation $y = \left| \frac{5x^2}{5x - 2} \right|$ and find in exact form the set of values of x for which

$$\left| \frac{5x^2}{5x - 2} \right| < 2.$$

[6]

Q27

9231/12 • October/November 2022 • Question 7

The curve C has equation $y = \frac{x^2 - x}{x + 1}$.

(a) Find the equations of the asymptotes of C . [3]

(b) Find the exact coordinates of the stationary points on C . [4]

(c) Sketch C , stating the coordinates of any intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left| \frac{x^2 - x}{x + 1} \right|$ and find in exact form the set of values of x for which

$$\left| \frac{x^2 - x}{x + 1} \right| < 6.$$

[5]

Q28

9231/11, 9231/12 • May/June 2023 • Question 6

The curve C has equation $y = \frac{x^2 + 2x - 15}{x - 2}$.

- (a) Find the equations of the asymptotes of C . [3]
- (b) Show that C has no stationary points. [3]
- (c) Sketch C , stating the coordinates of the intersections with the axes. [3]
- (d) Sketch the curve with equation $y = \left| \frac{x^2 - 2x - 15}{x - 2} \right|$. [2]
- (e) Find the set of values of x for which $\left| \frac{2x^2 + 4x - 30}{x - 2} \right| < 15$. [4]

Q29

9231/13 • May/June 2023 • Question 7

The curve C has equation $y = \frac{x^2 + 2x + 1}{x - 3}$.

- (a) Find the equations of the asymptotes of C . [3]
- (b) Find the coordinates of the turning points on C . [3]
- (c) Sketch C . [3]
- (d) Sketch the curves with equations $y = \left| \frac{x^2 + 2x + 1}{x - 3} \right|$ and $y^2 = \frac{x^2 + 2x + 1}{x - 3}$ on a single diagram, clearly identifying each curve. [4]

Q30

9231/11, 9231/13 • October/November 2023 • Question 7

The curve C has equation $y = f(x)$, where $f(x) = \frac{x^2 + 2}{x^2 - x - 2}$.

- (a) Find the equations of the asymptotes of C . [2]
- (b) Find the coordinates of any stationary points on C , giving your answers correct to 1 decimal place. [4]
- (c) Sketch C , stating the coordinates of any intersections with the axes. [3]
- (d) Sketch the curve with equation $y = \frac{1}{f(x)}$. [2]
- (e) Find the set of values for which $\frac{1}{f(x)} < f(x)$. [4]

Q31

9231/12 • October/November 2023 • Question 7

The curve C has equation $y = f(x)$, where $f(x) = \frac{x^2}{x+1}$.

(a) Find the equations of the asymptotes of C . [3]

(b) Find the coordinates of any stationary points on C . [2]

(c) Sketch C . [3]

(d) Find the coordinates of any stationary points on the curve with equation $y = \frac{1}{f(x)}$. [2]

(e) Sketch the curve with equation $y = \frac{1}{f(x)}$ and find, in exact form, the set of values for which

$$\frac{1}{f(x)} > f(x).$$

[6]

Q32

9231/11, 9231/12 • May/June 2024 • Question 6

The curve C has equation $y = \frac{x^2 + ax + 1}{x + 2}$, where $a > \frac{5}{2}$.

- (a) Find the equations of the asymptotes of C . [3]
- (b) Show that C has no stationary points. [4]
- (c) Sketch C , stating the coordinates of the point of intersection with the y -axis and labelling the asymptotes. [3]
- (d) (i) Sketch the curve with equation $y = \left| \frac{x^2 + ax + 1}{x + 2} \right|$. [2]
- (ii) On your sketch in part (i), draw the line $y = a$. [1]
- (iii) It is given that $\left| \frac{x^2 + ax + 1}{x + 2} \right| < a$ for $-5 - \sqrt{14} < x < -3$ and $-5 + \sqrt{14} < x < 3$.
Find the value of a . [2]

Q33

9231/13 • May/June 2024 • Question 6

The curve C has equation $y = \frac{x+1}{x^2+3}$.

- (a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote. [2]
- (b) Find the coordinates of any stationary points on C . [4]
- (c) Sketch C , stating the coordinates of the intersections with the axes. [3]
- (d) Sketch $y^2 = \frac{x+1}{x^2+3}$, stating the coordinates of the stationary points and the intersections with the axes. [4]

Q34

9231/11, 9231/13 • October/November 2024 • Question 6

The curve C has equation $y = \frac{4x^2 + x + 1}{2x^2 - 7x + 3}$.

(a) Find the equations of the asymptotes of C . [2]

(b) Find the coordinates of any stationary points on C . [4]

(c) Sketch C , stating the coordinates of any intersections with the axes. [5]

(d) Sketch the curve with equation $y = \left| \frac{4x^2 + x + 1}{2x^2 - 7x + 3} \right|$ and state the set of values of k for which $\left| \frac{4x^2 + x + 1}{2x^2 - 7x + 3} \right| = k$ has 4 distinct real solutions. [2]

Q35

9231/12 • October/November 2024 • Question 6

The curve C has equation $y = \frac{x^2 + 3}{x^2 + 1}$.

- (a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote. [2]
- (b) Show that $1 < y \leq 3$ for all real values of x . [4]
- (c) Find the coordinates of any stationary points on C . [2]
- (d) Sketch C , stating the coordinates of any intersections with the axes and labelling the asymptote. [3]
- (e) Sketch the curve with equation $y = \frac{x^2 + 1}{x^2 + 3}$ and find the set of values of x for which $\frac{x^2 + 1}{x^2 + 3} < \frac{1}{2}$. [4]

Q36

9231/11, 9231/12 • May/June 2025 • Question 7

The curve C has equation $y = \frac{2x^2 - 5x}{2x^2 - 7x - 4}$.

- (a) Find the equations of the asymptotes of C . [2]
- (b) Find the coordinates of any stationary points on C . [4]
- (c) Sketch C , stating the coordinates of the intersections with the axes. [3]
- (d) Sketch the curve with equation $y = \left| \frac{2x^2 - 5x}{2x^2 - 7x - 4} \right|$. [1]
- (e) Find in exact form the set of values of x for which $\left| \frac{2x^2 - 5x}{2x^2 - 7x - 4} \right| < \frac{1}{9}$. [5]

Q37

9231/13 • May/June 2025 • Question 6

The curve C has equation $y = \frac{x^2 + a}{x + a}$, where a is a positive constant.

- (a) Find the equations of the asymptotes of C . [3]
- (b) Find, in terms of a , the x -coordinates of the stationary points on C . [3]
- (c) Sketch C , stating the coordinates of any intersections with the axes. [3]
- (d) Sketch the curve with equation $y = \left| \frac{x^2 + a}{x + a} \right|$. [1]
- (e) Find the set of values of a for which $\left| \frac{x^2 + a}{x + a} \right| = a$ has two real solutions. [4]

Q38

9231/11, 9231/13 • October/November 2025 • Question 7

The curve C has equation $y = \frac{x+2}{x^2+3x+1}$.

(a) Find the equations of the asymptotes of C . [2]

(b) Show that C has no stationary points. [3]

(c) Sketch C , stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left| \frac{x+2}{x^2+3x+1} \right|$. [2]

(e) Find in exact form the set of values of x for which $\left| \frac{x+2}{x^2+3x+1} \right| > 2$. [4]

Q39

9231/12 • October/November 2025 • Question 7

The curve C has equation $y = \frac{x^2 + x + 1}{x + 1}$.

(a) Find the equations of the asymptotes of C . [3]

(b) Find the coordinates of any stationary points on C . [3]

(c) Sketch C . [3]

(d) Sketch the curve with equation $y = \frac{|x|^2 + |x| + 1}{|x| + 1}$. [2]

(e) Find, in exact form, the set of values of x for which $\frac{|x|^2 + |x| + 1}{|x| + 1} < 3$. [3]

Q40

9231/14 • October/November 2025 • Question 7

The curve C has equation $y = \frac{10x^2 - 11x - 18}{10x - 18}$.

(a) Find the equations of the asymptotes of C . [3]

(b) Show that C has no stationary points. [4]

(c) Sketch C , stating the coordinates of the points of intersection with the axes. [3]

(d) Sketch the curve with equation $y = \left| \frac{10x^2 - 11x - 18}{10x - 18} \right|$. [2]

(e) Given that $\left| \frac{10x^2 - 11x - 18}{10x - 18} \right| < 4$ for $\frac{-29 - \sqrt{4441}}{20} < x < p$ and $\frac{-29 + \sqrt{4441}}{20} < x < q$, find the values of p and q . [4]

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Written by Ali Hashir.

Answers available in the companion mark scheme booklet.



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