Further Math with Ali Hashir

Rational Functions

Topical Mark Schemes (CIE 9231)

Mark schemes corresponding to the topical questions booklet for this topic, compiled from official CIE 9231 Paper 1 marking schemes (2015–2025). Question numbers match the questions booklet.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Contents

Click a question number below to jump straight to its mark scheme. Every box is a live link.

For the best hyperlink experience, open this PDF in a dedicated reader – such as Adobe Acrobat, Apple Preview, or your browser's built-in PDF viewer – rather than a basic in-app or embedded preview, where links sometimes don't work.

Q1

9231/13 • May/June 2015 • Question 10

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Mark scheme.

$-\frac{1}{2}+\frac{(3x-1)^2}{2(x^2+1)}=\frac{-x^2-1+9x^2-6x+1}{2(x^2+1)}=\frac{8x^2-6x}{2(x^2+1)}=\frac{4x^2-3x}{x^2+1}$	**M1A1**
$\frac{9}{2}-\frac{(x+3)^2}{2(x^2+1)}=\frac{9x^2+9-x^2-6x-9}{2(x^2+1)}=\frac{8x^2-6x}{2(x^2+1)}=\frac{4x^2-3x}{x^2+1}$	**A1**
	(3)
$\frac{(3x-1)^2}{2(x^2+1)}\geqslant 0\Rightarrow y\geqslant -\frac{1}{2}$	**B1**
$\frac{(x+3)^2}{2(x^2+1)}\geqslant 0\Rightarrow y\leqslant \frac{9}{2}\Rightarrow -\frac{1}{2}\leqslant y\leqslant \frac{9}{2}$ (Not hence gets B1 only.)	**B1**
	(2)
Turning points occur at equality, so $\left(-3,\frac{9}{2}\right)$ and $\left(\frac{1}{3},-\frac{1}{2}\right)$	**B1B1**
	(2)
Asymptote is $y=4$.	**B1**
	(1)
Intersection with axes at $(0,0)$ and $\left(\frac{3}{4},0\right)$, Intersection with asymptote at $\left(-\frac{4}{3},4\right)$	**B1B1**
Correct graph.	**B1**
	(3)
	Total: 11

Q2

9231/11, 9231/12, 9231/13 • October/November 2015 • Question 8

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Mark scheme.

$y' = 0 \Rightarrow (x+1)(4x+k) - (2x^2+kx) \times 1 = 0$	**M1**
$\Rightarrow 4x^2 + (4+k)x + k - 2x^2 - kx = 0 \Rightarrow 2x^2 + 4x + k = 0$	**A1**
$B^2 - 4AC < 0 \Rightarrow$ no stationary points $\Rightarrow 16 - 8k < 0$	**M1A1**
$\Rightarrow k > 2$ for no stationary points.	**A1**
	(5)
When $k = 4$:	
Vertical asymptote: $x = -1$	**B1**
Oblique asymptote: $y = 2x + 2 - \dfrac{2}{x+1} \Rightarrow y = 2x + 2$	**M1A1**
Axes and asymptotes	**B1**
Each branch.	**B1B1**
	(6)
	Total: 11

Q3

9231/11, 9231/12 • May/June 2016 • Question 7

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Mark scheme.

Vertical asymptote is $x = 2$.	**B1**
$y = x + 2 + \dfrac{4}{x-2} \Rightarrow$ Oblique asymptote is $y = x + 2$.	**M1A1**
	(3)
$y = \dfrac{x^2}{x-2} \Rightarrow x^2 - yx + 2y = 0$	**B1**
Quadratic has no real roots (i.e. no points on C) if $\Delta < 0 \Rightarrow y^2 - 8y < 0$	**M1**
$\Rightarrow y(y-8) < 0 \Rightarrow 0 < y < 8$. (AG) Correct inequality	**M1A1**
	(4)
Axes and asymptotes.	**B1**
Each branch, showing $(0, 0)$ and $(4, 8)$.	**B1B1**
(Deduct at most 1 mark for poor forms at infinity and/or missing coordinates.)	
	(3)
	Total: 10

Q4

9231/13 • May/June 2016 • Question 5

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Mark scheme.

$\frac{dy}{dx} = \frac{(x^2-9)-2x(x+2)}{(x^2-9)^2}$	**B1**
$= \frac{-x^2-4x-9}{(x^2-9)^2} = \frac{-(x+2)^2-5}{(x^2-9)^2} \Rightarrow \frac{dy}{dx} < 0$	**M1A1**
	(3)
Asymptotes: $x = \pm 3$; $y = 0$	**B1B1**
	(2)
Sketch: Axes and asymptotes; Outside branches	**B1B1**
Middle branch, showing $\left(0, -\frac{2}{9}\right)$ and $(-2, 0)$.	**B1**
(Deduct at most 1 mark for poor forms at infinity.)	
	(3)
	Total: 8

Q5

9231/11, 9231/12 • May/June 2017 • Question 9

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Mark scheme.

$x = 1$	**B1**
$y = 2 - x$	**M1A1**
	(3)
$y' = -1 + 4(1-x)^{-2} = 0$	**M1**
$\Rightarrow x = -1, 3$	**A1**
Turning points are $(-1, 5)$ and $(3, -3)$	**A1**
	(3)
$(0, 6)$	**B1**
$y = 0 \Rightarrow x^2 - 3x + 6$; $\Delta = 9 - 24 \Rightarrow$ No intersection with x-axis	**B1**
	(2)
Asymptotes correct	**B1 FT**
Each branch	**B1B1**
	(3)
	Total: 11

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Q6

9231/11, 9231/12, 9231/13 • October/November 2017 • Question 9

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Mark scheme.

Degree of numerator $<$ degree of denominator $\Rightarrow y = 0$ is horizontal asymptote.	**B1**
$(x+1)(x-2) = 0 \Rightarrow x = -1$ and $\Rightarrow x = 2$ are vertical asymptotes.	**B1**
	(2)
$yx^2 - (y+3)x + 9 - 2y = 0$	**M1**
No points on C if $(y+3)^2 - 4y(9-2y) < 0$	**M1**
$\Rightarrow 9y^2 - 30y + 9 < 0 \Rightarrow 3y^2 - 10y + 3 < 0$	**A1**
$\Rightarrow (3y-1)(y-3) < 0 \Rightarrow \frac{1}{3} < y < 3 \quad$ (AG)	**A1**
	(4)
$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 3(x^2 - x - 2) - (3x-9)(2x-1) = 0$	**B1**
$\Rightarrow \ldots \Rightarrow (x-1)(x-5) = 0$	**B1**
$\Rightarrow$ Turning points are $(1,3)$ and $\left(5, \frac{1}{3}\right)$.	**B1**
	(3)
Axes, asymptotes and points on axes $(0, 4.5)$, $(3, 0)$.	**B1**
RH branch; Other two branches	**B1B1**
	(3)
	Total: 12

Q7

9231/11, 9231/12 • May/June 2018 • Question 6

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Mark scheme.

Vertical asymptote is $x = -b$.	**B1**
$x^2+b = (x+b)(x-b)+b^2+b$ or $x+b\overline{)x^2+0x+b}$ by inspection or long division	**M1**
Thus the oblique asymptote is $y = x - b$	**A1**
	(3)
If $y = 0$ then $x^2 + b = 0$ which has no real root. Must refer to $b > 0$ OE	**B1**
	(1)
$\dfrac{dy}{dx} = \dfrac{2x(x+b)-(x^2+b)}{(x+b)^2} = 0 \Rightarrow x^2 + 2bx - b = 0$	**M1**
Or differentiating $y = x - b + \dfrac{b^2+b}{x+b}$ and setting $\dfrac{dy}{dx} = 0$ gives $1 - \dfrac{b^2+b}{(x+b)^2} = 0$	
$b^2 + b > 0$. Therefore there are two stationary points on C. Use discriminant or $(x + b)^2$ to show two stationary points	**A1**
	(2)
Sketch: intersection $(0, 1)$ given and asymptotes drawn	**B1**
Each branch correct. Penalise at most one mark for poor forms at infinity	**B1B1**
	(3)
	Total: 9

Q8

9231/13 • May/June 2018 • Question 4

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Mark scheme.

$(-6,0),(-1,0)$ States points of intersection with x-axis	**B1**
$(0,-3)$ States y-intercept	**B1**
	(2)
One asymptote is $x=2$.	**B1**
$y=x+9+\dfrac{24}{x-2} \Rightarrow$ other asymptote is $y=x+9$. By inspection or long division. A0 if error in division	**M1A1**
	(3)
Sketches axes and asymptotes, labelled or to scale	**B1**
Upper branch correctly located and orientated.	**B1**
Lower branch correctly located and orientated. Penalise at most 1 mark for poor forms at infinity	**B1**
	(3)
	Total: 8

Q9

9231/11, 9231/13 • October/November 2018 • Question 6

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Mark scheme.

Vertical asymptote is $x = -1$.	**B1**
$x^2+ax-1 = (x+1)(x+a-1)-a$ or $x+1\overline{)x^2+ax-1}$ giving quotient $x+a-1$ (By inspection or long division.)	**M1**
Thus the oblique asymptote is $y = x + a - 1$	**A1**
	(3)
$a^2 + 4 > 0$	**B1**
	(1)
$\dfrac{(x+1)(2x+a)-(x^2+ax-1)}{(x+1)^2} = 0 \Rightarrow x^2 + 2x + a + 1 = 0$	**M1**
Discriminant $= 4 - 4(a + 1) < 0$	
Therefore there are no stationary points on C.	**A1**
	(2)
[Sketch: curve with vertical asymptote $x = -1$ and oblique asymptote $y = x + a - 1$, two branches, correct y-intercept and asymptotes drawn.] (Correct y-intercept and asymptotes drawn.)	**B1**
Each branch correct	**B1B1**
	(3)
	Total: 9

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Q10

9231/12 • October/November 2018 • Question 9

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Mark scheme.

$y = 5 - \dfrac{4}{x^2+x+1}$ (Alt method: Finding limit)	**M1**
As $x \to \pm\infty$, $y \to 5$ $\therefore y = 5$ CAO	**A1**
	(2)
$yx^2 + yx + y = 5x^2 + 5x + 1$	**B1**
$\Rightarrow (y-5)x^2 + (y-5)x + (y-1) = 0$ (Forms quadratic equation in x.)	
For real x, $(y-5)^2 - 4(y-5)(y-1) \geqslant 0$ (condone $>$) (Uses discriminant)	**M1**
$\Rightarrow (y-5)(3y+1) \leqslant 0$ (Factorising)	**M1**
$\Rightarrow -\dfrac{1}{3} \leqslant y < 5$, because $y = 5$ is an asymptote (www) (Explaining strict upper inequality (AG))	**A1**
	(4)
$y' = 0 \Rightarrow (x^2+x+1)(10x+5) - (5x^2+5x+1)(2x+1) = 0$ (Differentiates and equates to 0.)	**M1**
$\Rightarrow 4(2x+1) = 0 \Rightarrow x = -\dfrac{1}{2}, y = -\dfrac{1}{3}$	**A1**
	(2)
[Sketch: curve with minimum at $\left(-\frac{1}{2}, -\frac{1}{3}\right)$, horizontal asymptote $y = 5$, positive y-intercept at $(0, 1)$.] (Positive y-intercept at $(0, 1)$, FT dep on minimum point from (iii).)	**B1FT**
Correct asymptote and completely correct graph.	**B1**
	(2)
	Total: 10

Q11

9231/11, 9231/12 • May/June 2019 • Question 10

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Mark scheme.

$x = -5$ and $y = a$	**B1B1**
	(2)
$x^2 + (a+10)x + 5a + 26 = (x+5)(x+a+5) + 1$	**M1**
oblique asymptote is $y = x + a + 5$	**A1**
	(2)
$x^2 + 10x + 5a + 26 = 0$	**M1**
$10^2 - 4(5a+26) = -4 - 20a < 0$ so no intersection point	**A1**
	(2)
$(x+5)(2x+a+10) - x^2 - ax - 10x - 5a - 26 = 0$	**M1**
$x^2 + 10x + 24 = 0$	**A1**
Stationary points are $(-4, a+2)$ and $(-6, a-2)$	**A1**
	(3)
[Sketch: asymptotes drawn, intersection correct]	**B1**
[Sketch: C_1 correct]	**B1**
[Sketch: C_2 correct]	**B1**
	(3)
	Total: 12

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Q12

9231/13 • May/June 2019 • Question 6

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Mark scheme.

$x = \dfrac{1}{k}$	**B1**
$y = \dfrac{(kx-1)(k^{-1}x + k^{-2}) + k^{-2}}{kx - 1}$	**M1**
Oblique asymptote is $y = k^{-1}x + k^{-2}$	**A1**
	(3)
$y' = \dfrac{2x(kx-1) - kx^2}{(kx-1)^2} = 0 \Rightarrow kx^2 - 2x = 0$	**M1**
$x = 0, 2k^{-1}$	**A1**
$(0,0), (2k^{-1}, 4k^{-2})$	**A1**
	(3)
[Sketch: axes and asymptotes correct]	**B1**
[Sketch: upper branch correct]	**B1**
[Sketch: lower branch correct]	**B1**
	(3)
	Total: 9

Q13

9231/11, 9231/12, 9231/13 • October/November 2019 • Question 4

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Mark scheme.

$x^2+1=(ax+b)(2x+1)+c$	**M1**
Uses that $2x+1$ is the quotient.	
$\Rightarrow a=\frac{1}{2}, b=-\frac{1}{4}$	**A1A1**
	(3)
$x=\frac{1}{2}$	**B1FT**
	(1)
Sketch showing intersection $(0,-4)$ and both asymptotes drawn.	**B1**
Left branch correct.	**B1**
Right branch correct.	**B1FT**
Deduct at most one mark for poor forms at infinity.	
	(3)
	Total: 7

Q14

9231/11, 9231/12 • May/June 2020 • Question 1

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Mark scheme.

Sketch: correct axes and asymptotes	**B1**
Sketch: branches correct	**B1**
	(2)
Sketch: **B1 FT** from sketch in part (a)	**B1 FT**
$\frac{ax}{x+7} = \frac{a}{2}$ or $\frac{ax}{x+7} = -\frac{a}{2}$	**M1**
$x = 7$ and $x = -\frac{7}{3}$	**A1**
$x < -7$, $-7 < x < -\frac{7}{3}$, $x > 7$	**A1**
	(4)
	Total: 6

Q15

9231/11, 9231/12 • May/June 2020 • Question 3

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Mark scheme.

$x = -\dfrac{1}{2}$	**B1**
$y = \dfrac{(2x+1)(\frac{1}{2}x - \frac{1}{4}) + \frac{1}{4}}{2x+1} = \frac{1}{2}x - \frac{1}{4} + \dfrac{1}{4(2x+1)}$	**M1**
$y = \frac{1}{2}x - \frac{1}{4}$	**A1**
	(3)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2} - \dfrac{1}{2(2x+1)^2} = 0 \Rightarrow (2x+1)^2 = 1$	**M1**
$x = 0, -1$	**A1**
$(0,0), (-1,-1)$	**A1**
	(3)
Sketch: correct axes and asymptotes	**B1**
Sketch: upper branch correct	**B1**
Sketch: lower branch correct	**B1**
	(3)
	Total: 9

Q16

9231/13 • May/June 2020 • Question 6

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Mark scheme.

$x = \frac{3}{2}$	**B1**
$-2x^2 + x + 10 = (2x-3)(-x-1) + 7 \Rightarrow y = -x - 1$	**M1A1**
	(3)
$\frac{dy}{dx} = \frac{(2x-3)(1-4x) - 2(10+x-2x^2)}{(2x-3)^2}$	**M1**
$4x^2 - 12x + 23 = 0 \left(\text{or } \frac{dy}{dx} = -1 - \frac{14}{(2x-3)^2}\right)$	**A1**
$12^2 - 4(4)(23) = -224 < 0$ (or $y' < 0$) $\Rightarrow$ No turning points.	**A1**
	(3)
Sketch: correct axes and asymptotes	**B1**
Sketch: branches correct	**B1**
$\left(0, -\frac{10}{3}\right), (-2, 0), \left(\frac{5}{2}, 0\right)$	**B1**
	(3)
Sketch: **B1 FT** for their sketch in (c)	**B1 FT**
Sketch: correct shape at infinity	**B1**
$\frac{10+x-2x^2}{2x-3} = 4$ or $\frac{10+x-2x^2}{2x-3} = -4$; $2x^2+7x-22=0$ or $2x^2-9x+2=0$	**M2**
$x = -\frac{11}{2}, 2$ or $x = \frac{1}{4}\left(9 \pm \sqrt{65}\right)$	**A1**
$-\frac{11}{2} < x < \frac{1}{4}\left(9-\sqrt{65}\right)$ and $2 < x < \frac{1}{4}\left(9+\sqrt{65}\right)$	**A1 FT**
	(6)
	Total: 15

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Q17

9231/11, 9231/13 • October/November 2020 • Question 6

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Mark scheme.

$x = 1$	**B1**
States vertical asymptote.	
$x^2 + x - 1 = (x-1)(x+2) + 1 \Rightarrow y = x + 2$	**M1A1**
Finds oblique asymptote.	
	(3)
$yx - y = x^2 + x - 1 \Rightarrow x^2 + (1-y)x + y - 1 = 0$	**M1A1**
Forms quadratic in x.	
$(1-y)^2 - 4(y-1) < 0 \Rightarrow y^2 - 6y + 5 < 0$	**M1**
Uses that discriminant is negative if there are no values of x.	
$1 < y < 5$	**A1**
AG	
	(4)
[Sketch: axes and asymptotes correct; branches correct.]	**B1B1**
$(0, 1)$, $\left(-\frac{1}{2} + \frac{1}{2}\sqrt{5}, 0\right)$, $\left(-\frac{1}{2} - \frac{1}{2}\sqrt{5}, 0\right)$	**B1**
States coordinates of intersections with axes, can be labelled on graph. Accept $(0.618, 0)$ and $(-1.62, 0)$.	
	(3)
[Sketch of $y = \lvert C \rvert$: FT from sketch in (c) with both branches; correct shape at extremities.]	**B1FT** **B1**
	(2)
	Total: 12

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Q18

9231/12 • October/November 2020 • Question 6

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Mark scheme.

[Sketch: asymptotes labelled; correct position and shape, not too truncated.]	**B1B1**
	(2)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{2a(x-a)}{(x-2a)^3} = 0$	**M1**
Sets $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.	
Solves for x.	**M1**
Since $x \neq 2a$, stationary point is $(a, 0)$.	**A1**
	(3)
$(x-a)^2 = (x-a)(x-2a)$ or $(x-a)^2 = -(x-a)(x-2a)$	**M1**
Finds a critical point.	
$x = a$ and $\frac{3}{2}a$ or $(a, 0)$ or $(\frac{3}{2}a, 1)$	**A1**
Both x values or one correct point.	
$(a, 0)$ and $(\frac{3}{2}a, 1)$	**A1**
	(3)
[Sketch: axes and asymptotes; C_3 FT from their sketch in part (a), clearly identified; correct shape of C_2, clearly identified; relative positions and intersections correct.]	**B1** **B1FT** **B1 B1**
$x \leqslant \frac{3}{2}a$	**B1**
Accept algebraic method.	
	(5)
	Total: 13

Q19

9231/11, 9231/12 • May/June 2021 • Question 7

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Mark scheme.

$x = -1$	**B1**
$y = \dfrac{x(x+1)+9}{x+1} = x + \dfrac{9}{x+1}$	**M1**
$y = x$	**A1**
	(3)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - 9(x+1)^{-2} = 0 \Rightarrow (x+1)^2 = 9$	**M1 A1**
$(2, 5)$	**A1**
$(-4, -7)$	**A1**
	(4)
Axes labelled and correct asymptotes drawn.	**B1**
Upper branch with $(0, 9)$ stated or shown on diagram.	**B1**
Lower branch correct and good approach to asymptotes throughout, no extra branches.	**B1**
	(3)
FT from sketch in (c) with asymptotes shown.	**B1 FT**
$x^2 + x + 9 = \dfrac{13}{2}(x+1)$ or $x^2 + x + 9 = -\dfrac{13}{2}(x+1)$ (may state that $x^2 + x + 9 = -\frac{13}{2}(x+1)$ has no real solutions since $7 > \frac{13}{2}$)	**M1 M1**
$x^2 - \dfrac{11}{2}x + \dfrac{5}{2} = 0$ or $x^2 + \dfrac{15}{2}x + \dfrac{31}{2} = 0$	
$x = \dfrac{1}{2}, 5$	**A1**
$x < \dfrac{1}{2}$ and $x > 5$.	**A1**
	(5)
	Total: 15

Q20

9231/13 • May/June 2021 • Question 7

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Mark scheme.

$x = \dfrac{1-\sqrt{5}}{2},\ x = \dfrac{1+\sqrt{5}}{2}$	**B1**
$y = -1$	**B1**
	(2)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(1+x-x^2)(2x-1)-(x^2-x-3)(1-2x)}{(1+x-x^2)^2}$	**M1**
$(2x-1)(-2) = 0$	**M1**
$\left(\dfrac{1}{2}, -\dfrac{13}{5}\right)$	**A1**
	(3)
Both axes labelled and correct asymptotes shown.	**B1**
Correct shape and position, with all asymptotic behaviour clear.	**B1**
States exact coordinates of intersections with axes: $\left(\frac{1}{2}+\frac{1}{2}\sqrt{13}, 0\right)$, $\left(\frac{1}{2}-\frac{1}{2}\sqrt{13}, 0\right)$, $(0,-3)$	**B1**
	(3)
FT from sketch in (c).	**B1 FT**
Correct shape as x tends to infinity and intersections with x axis.	**B1**
$\dfrac{x^2-x-3}{1+x-x^2} = 3$ or $\dfrac{x^2-x-3}{1+x-x^2} = -3$; $4x^2-4x-6=0$ or $-2x^2+2x=0$	**M2**
$x = \dfrac{1}{2}+\dfrac{1}{2}\sqrt{7},\ x = \dfrac{1}{2}-\dfrac{1}{2}\sqrt{7},\ x=0,\ x=1$ (must be exact)	**A1**
$x < \dfrac{1}{2}-\dfrac{1}{2}\sqrt{7},\ 0<x<1,\ x > \dfrac{1}{2}+\dfrac{1}{2}\sqrt{7}$ (must be three distinct regions and strict inequalities)	**A1 FT**
	(6)
	Total: 14

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Q21

9231/11, 9231/13 • October/November 2021 • Question 7

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Mark scheme.

$x = 1,\ x = -1$	**B1**
Vertical asymptotes.	
$y = 0$	**B1**
Horizontal asymptote.	
	(2)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(4-4x^2)(4)-(4x+5)(-8x)}{(4-4x^2)^2}$	**M1**
Finds $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.	
$16x^2 + 40x + 16 = 0$	**M1**
Sets equal to 0 and forms equation.	
$(-2, \frac{1}{4}),\ (-\frac{1}{2}, 1)$	**A1A1**
	(4)
Axes and asymptotes.	**B1**
Correct shape and position.	**B1**
$(-\frac{5}{4}, 0),\ (0, \frac{5}{4})$	**B1**
States coordinates of intersections with axes.	
	(3)
FT from sketch in part (c).	**B1FT**
Correct shape at infinity.	**B1**
$\dfrac{4x+5}{4-4x^2} = \dfrac{5}{4}$ or $\dfrac{4x+5}{4-4x^2} = -\dfrac{5}{4}$	**M2**
$5x^2 + 4x = 0$ or $5x^2 - 4x - 10 = 0$	

Finds critical points, award M1 for each case.	
$x = -\frac{4}{5}$, $x = 0$ or $x = \frac{2}{5} - \frac{3}{5}\sqrt{6}$, $x = \frac{2}{5} + \frac{3}{5}\sqrt{6}$	**A1**
A0 if -1.07, 1.87	
$\frac{2}{5} - \frac{3}{5}\sqrt{6} < x < -\frac{4}{5}$, $0 < x < \frac{2}{5} + \frac{3}{5}\sqrt{6}$, $x \neq \pm 1$	**A1FT**
Condone exclusion of $x = \pm 1$ from the range.	
	(6)

Total: 15

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Q22

9231/12 • October/November 2021 • Question 6

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Mark scheme.

$x = 3$	**B1**
States vertical asymptote.	
$y = x + 3 + \dfrac{9}{x-3}$ leading to $y = x + 3$	**M1A1**
Finds oblique asymptote.	
	(3)
$yx - 3y = x^2$ leading to $x^2 - yx + 3y = 0$	**M1A1**
Forms quadratic in x.	
$y^2 - 4(3y) < 0$ leading to $y^2 - 12y < 0$	**M1**
Uses that discriminant is negative.	
$0 < y < 12$	**A1**
AG	
	(4)
Axes and asymptotes.	**B1**
Branches correct.	**B1**
	(2)
FT from sketch in (c).	**B1FT**
Correct shape at infinity.	**B1**
Correct shape of $y = \lvert x \rvert - 3$.	**B1**
Correct intercepts with axes (may be seen on graph).	**B1**
	(4)

$c \leqslant -3$	**B1**
	(1)
	Total: 14

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Q23

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Mark scheme.

$-3 < 0$	**M1 A1**
$y = 2$	**B1**
	(3)
$\dfrac{dy}{dx} = \dfrac{(x^2+x+1)(4x-1)-(2x^2-x-1)(2x+1)}{(x^2+x+1)^2}$	**M1**
$3x^2 + 6x = 0$	**M1**
$(0,-1) \quad (-2,3)$	**A1 A1**
	(4)
Sketch: axes and asymptote shown.	**B1**
Sketch: correct shape and position.	**B1**
States coordinates of intersections with axes: $(1,0)$, $\left(-\frac{1}{2},0\right)$, $(0,-1)$.	**B1**
	(3)
Sketch of $\lvert y\rvert$-curve follows through from part (c), with cusps on the x axis.	**B1 FT**
$0 < k < 1$	**B1**
	(2)
	Total: 12

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Q24

9231/13 • May/June 2022 • Question 1

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Mark scheme.

Sketch: axes and asymptotes.	**B1**
Sketch: branches correct.	**B1**
	(2)
Sketch follows through from part (a), symmetrical about $x = 0$.	**B1 FT**
Correct shape at $x = 0$ (reflection not turning point).	**B1**
$\dfrac{\lvert x\rvert + 1}{\lvert x\rvert - 1} = -2$ leading to $\lvert x\rvert = \frac{1}{3}$	**M1**
$-1 < x < -\frac{1}{3}$, $\frac{1}{3} < x < 1$	**A1**
	(4)
	Total: 6

Q25

9231/13 • May/June 2022 • Question 3

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Mark scheme.

$x = 1$	**B1**
$y = \dfrac{(x-1)(ax+a+1)+a}{x-1} = ax + a + 1 + \dfrac{a}{x-1}$	**M1**
$y = ax + a + 1$	**A1**
	(3)
$ax^2 + x - 1 = yx - y$ leading to $ax^2 - (y-1)x + (y-1) = 0$	**M1 A1**
$(y-1)^2 - 4a(y-1) < 0$	**M1**
$1 < y < 1 + 4a$	**A1**
	(4)
Asymptotes identified. Follow through from part (a).	**B1 FT**
Two branches in correct position relative to correct asymptotes and of correct shape. Good asymptotic behaviour.	**B1**
Lower branch in all four quadrants from correct asymptotes.	**B1**
	(3)
	Total: 10

Q26

9231/11, 9231/13 • October/November 2022 • Question 7

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Mark scheme.

$x = \frac{2}{5}$	**B1**
$y = \dfrac{(5x-2)(x+\frac{2}{5})+\frac{4}{5}}{5x-2}$ leading to $y = x + \frac{2}{5}$	**M1A1**
	(3)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(5x-2)(10x)-(5x^2)(5)}{(5x-2)^2}$	**M1**
$5x^2 - 4x = 0$	**M1**
$(0,0),\ \left(\frac{4}{5}, \frac{8}{5}\right)$	**A1A1**
	(4)
Axes and asymptotes.	**B1**
Correct upper branch and asymptotic behaviour.	**B1**
Correct lower branch.	**B1**
	(3)
FT from sketch in part (c).	**B1FT**
Correct shape as x tends to infinity.	**B1**
$\dfrac{5x^2}{5x-2} = 2$ or $\dfrac{5x^2}{5x-2} = -2$; $5x^2 - 10x + 4 = 0$ or $5x^2 + 10x - 4 = 0$	**M2**
$x = -1 - \frac{3}{5}\sqrt{5},\ x = -1 + \frac{3}{5}\sqrt{5}$ or $x = 1 - \frac{1}{5}\sqrt{5},\ x = 1 + \frac{1}{5}\sqrt{5}$	**A1**
$-1 - \frac{3}{5}\sqrt{5} < x < -1 + \frac{3}{5}\sqrt{5},\ 1 - \frac{1}{5}\sqrt{5} < x < 1 + \frac{1}{5}\sqrt{5}$	**A1FT**
	(6)
	Total: 16

Q27

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Mark scheme.

$x = -1$	**B1**
$y = \dfrac{(x+1)(x-2)+2}{x+1}$	**M1**
$y = x - 2$	**A1**
	(3)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - 2(x+1)^{-2} = 0 \Rightarrow (x+1)^2 = 2$	**M1A1**
$\left(-1+\sqrt{2}, -3+2\sqrt{2}\right),\ \left(-1-\sqrt{2}, -3-2\sqrt{2}\right)$	**A1A1**
	(4)
Axes and asymptotes labelled.	**B1**
Upper branch with $(0,0)$ and $(1,0)$ stated or clear on scale.	**B1**
Lower branch correct and all asymptotic approaches correct.	**B1**
	(3)
FT from sketch in part (c).	**B1FT**
$\dfrac{x^2-x}{x+1} = 6$ or $\dfrac{x^2-x}{x+1} = -6$; $x^2 - 7x - 6 = 0$ or $x^2 + 5x + 6 = 0$	**M2**
$x = -3,\ -2,\ \frac{7}{2} - \frac{1}{2}\sqrt{73},\ \frac{7}{2} + \frac{1}{2}\sqrt{73}$	**A1**
$-3 < x < -2$ and $\frac{7}{2} - \frac{1}{2}\sqrt{73} < x < \frac{7}{2} + \frac{1}{2}\sqrt{73}$	**A1**
	(5)
	Total: 15

Q28

9231/11, 9231/12 • May/June 2023 • Question 6

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Mark scheme.

$x = 2$	**B1**
$x^2 + 2x - 15 = (x-2)(x+4) - 7 \Rightarrow y = x + 4$	**M1A1**
	(3)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(x-2)(2x+2) - (x^2+2x-15)}{(x-2)^2}$	**M1**
$x^2 - 4x + 11 = 0 \quad (\text{or } \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 + \dfrac{7}{(x-2)^2})$	**A1**
$4^2 - 4(1)(11) = -28 < 0$ There are no turning points.	**A1**
	(3)
Axes and asymptotes.	**B1**
Branches correct.	**B1**
$(0, 7.5), (-5, 0), (3, 0)$	**B1**
	(3)
FT from sketch in (c).	**B1FT**
Correct shape at infinity and on x axis.	**B1**
	(2)
$\dfrac{x^2+2x-15}{x-2} = \dfrac{15}{2}$ or $\dfrac{x^2+2x-15}{x-2} = -\dfrac{15}{2}$; $\quad x^2 - \frac{11}{2}x = 0$ or $x^2 + \frac{19}{2}x - 30 = 0$	**M2**
$x = 0, \dfrac{11}{2}$ or $x = -12, \dfrac{5}{2}$	**A1**
$-12 < x < 0$ or $\dfrac{5}{2} < x < \dfrac{11}{2}$	**A1FT**
	(4)
	Total: 15

Q29

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Mark scheme.

$x = 3$	**B1**
$y = \dfrac{(x-3)(x+5)+16}{x-3} = x + 5 + \dfrac{16}{x-3}$	**M1**
$y = x + 5$	**A1**
	(3)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - \dfrac{16}{(x-3)^2} = 0 \Rightarrow (x-3)^2 = 16$	**M1**
$x = -1, 7$	**A1**
$(-1, 0), (7, 16)$	**A1**
	(3)
Axes and labelled asymptotes.	**B1FT**
Upper branch correct.	**B1**
Lower branch correct and no additional branches.	**B1**
	(3)
Clear labels, axes and their vertical asymptote.	**B1FT**
$y = \left\lvert \dfrac{x^2+2x+1}{x-3} \right\rvert$ correct, FT from their sketch in (c).	**B1FT**
Upper branch of $y^2 = \dfrac{x^2+2x+1}{x-3}$ (positive square root).	**B1**
Lower branch of $y^2 = \dfrac{x^2+2x+1}{x-3}$ (negative square root). FT from previous mark.	**B1FT**
	(4)
	Total: 13

Q30

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Mark scheme.

$x = -1,\ x = 2$	**B1**
$y = 1$ ((a) Total: 2)	**B1**
$\dfrac{dy}{dx} = \dfrac{(x^2 - x - 2)(2x) - (x^2 + 2)(2x - 1)}{(x^2 - x - 2)^2}$	**M1***
$x^2 + 8x - 2 = 0$	**DM1**
$(-8.2, 0.9),\ (0.2, -0.9)$. ((b) Total: 4)	**A1A1**
Axes and all three asymptotes.	**B1**
Correct shape and position, crossing horizontal asymptote.	**B1**
States $(0, -1)$ coordinates of intersection with axes, may be seen on diagram. ((c) Total: 3)	**B1**
FT from sketch in (c)	**B1FT**
All correct. ((d) Total: 2)	**B1**
$\dfrac{x^2 + 2}{x^2 - x - 2} = 1$ or $\dfrac{x^2 + 2}{x^2 - x - 2} = -1$	**M2**
$x + 4 = 0$ or $2x^2 - x = 0$	
$x = -4$ or $x = 0,\ x = \frac{1}{2}$	**A1**
$-4 < x < -1,\ 0 < x < \frac{1}{2},\ x > 2$ ((e) Total: 4)	**B1**
	Total: 15

Q31

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Mark scheme.

$x = -1$	**B1**
$y = \dfrac{(x+1)(x-1)+1}{x+1}$	**M1**
$y = x - 1$ ((a) Total: 3)	**A1**
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2+2x}{(x+1)^2} = 0$	**M1**
$(0,0)$, $(-2,-4)$ ((b) Total: 2)	**A1**
Axes and asymptotes.	**B1**
Left branch correct.	**B1**
Right branch correct. ((c) Total: 3)	**B1**
$\left(-2, -\frac{1}{4}\right)$ ((d) Total: 2)	**B1B1**
Left branch correct.	**B1**
Right branch correct.	**B1**
$\dfrac{x^2}{x+1} = 1$ or $\dfrac{x^2}{x+1} = -1$	**M2**
$x^2 - x - 1 = 0$	
$x = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ or $x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$	**A1**
$x < -1,\ \frac{1}{2} - \frac{1}{2}\sqrt{5} < x < \frac{1}{2} + \frac{1}{2}\sqrt{5},\ x \neq 0$ ((e) Total: 6)	**B1**
	Total: 16

Q32

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Mark scheme.

$x = -2$	**B1**
$y = \dfrac{(x+2)(x+a-2)-2a+5}{x+2} = x+a-2+\dfrac{5-2a}{x+2}$	**M1**
$y = x+a-2$	**A1**
	(3)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(x+2)(2x+a)-(x^2+ax+1)}{(x+2)^2}$	**M1** Differentiates.
$x^2+4x+2a-1=0 \quad \left(\text{or } \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1+\dfrac{2a-5}{(x+2)^2}\right)$	**A1** Forms quadratic equation or simplifies $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Not from wrong working.
$16-4(2a-1) = 20-8a < 0$ (or $y' > 0$) $\Rightarrow$ No stationary points	**M1 A1**
Consideration of discriminant or sign of y' with correct conclusion.	
	(4)
[Graph: branches of C approaching asymptotes $x=-2$ and $y=x+a-2$, correct shape either side of vertical asymptote]	**B1**
Axes and asymptotes labelled.	
B1	
Branches correct. (Asymptotes may cross above, on or below the x-axis.)	
$\left(0,\ \frac{1}{2}\right)$	**B1**
May be seen on their diagram.	

	(3)
[Graph: modulus curve with cusps on the x-axis and branch reflected in the x-axis where original curve is negative, approaching vertical asymptote from both sides]	
FT from their attempt in part (c).	**B1**
Everything correct (approach to vertical asymptote and cusps correct).	**B1**
	(2)
[Line $y = a$ drawn on sketch from (d)(i)]	**B1**
	(1)
$\dfrac{x^2+ax+1}{x+2} = a$ or $\dfrac{x^2+ax+1}{x+2} = -a$	**M1**
Or direct use of $x = \pm 3$	
$x^2 + 1 - 2a = 0$ or $x^2 + 2ax + 1 + 2a = 0$	
$-a - \sqrt{a^2-2a-1} < x < -\sqrt{2a-1},\ -a + \sqrt{a^2-2a-1} < x < \sqrt{2a-1}$	
$a = 5$	**A1**
	(2)
	Total: 15

Q33

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Mark scheme.

$x^2 + 3 = 0$ has no real roots.	**B1**
OE	
$y = 0$	**B1**
Horizontal asymptote.	
	(2)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(x^2+3) - (x+1)(2x)}{(x^2+3)^2}$	**M1** **Finds** $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.
$x^2 + 2x - 3 = 0$	**M1**
Sets equal to 0 and simplifies to quadratic equation.	
$\left(-3,\ -\frac{1}{6}\right),\ \left(1,\ \frac{1}{2}\right)$	**A1 A1**
	(4)
[Graph: bell-shaped curve approaching $y = 0$ from below as $x \to -\infty$, peak near $x = 1$, approaching $y = 0$ from above as $x \to \infty$]	
Axes and approach to asymptote.	**B1**
Correct smooth shape and position.	**B1**
$(-1, 0),\ \left(0,\ \frac{1}{3}\right)$	**B1**
States coordinates of intersections with axes, may be shown on their graph.	
	(3)
[Graph: $y^2 = \ldots$ curve, cusp at $(-1, 0)$, two branches symmetric about x-axis]	
Sections for $x \geqslant 0$ and $x < -1$ correct	**B1**
$-1 \leqslant x < 0$ correct.	**B1**

$(-1, 0),\ \left(0,\ \pm\frac{1}{\sqrt{3}}\right)$	**B1 States coordinates of intersections with axes, may be shown on their graph.**
$\left(1,\ \pm\frac{1}{\sqrt{2}}\right)$	**B1 May be shown on their graph.**
	(4)
	Total: 13

Q34

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Mark scheme.

$x = \frac{1}{2},\ x = 3$	**B1**
$y = 2$ ((a) Total: 2)	**B1**
$\frac{dy}{dx} = \frac{(2x^2 - 7x + 3)(8x + 1) - (4x^2 + x + 1)(4x - 7)}{(2x^2 - 7x + 3)^2}$	**M1**
$-3x^2 + 2x + 1 = 0$	**M1**
$\left(-\frac{1}{3}, \frac{1}{5}\right),\ (1, -3)$ ((b) Total: 4)	**A1A1**
[Sketch: axes and asymptotes, clear identification (label or clear intersection with axes at correct place).]	**B1**
[$x > 3$ correctly approaching asymptotes, not too truncated.]	**B1**
[$\frac{1}{2} < x < 3$ correct.]	**B1**
[$x < \frac{1}{2}$ correct.]	**B1**
$\left(0, \frac{1}{3}\right)$ ((c) Total: 5)	**B1**
[Sketch: FT from sketch in (c). At least two branches.]	**B1FT**
$k > 3$ ((d) Total: 2)	**B1**
	Total: 13

Q35

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Mark scheme.

$x^2 + 1 = 0$ has no real roots.	**B1**
$y = 1$ ((a) Total: 2)	**B1**
$yx^2 + y = x^2 + 3 \Rightarrow (1 - y)x^2 - y + 3 = 0$	**M1A1**
$-4(1 - y)(3 - y) \geq 0$	**M1**
$1 < y \leq 3$ ((b) Total: 4)	**A1**
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(x^2 + 1)(2x) - (x^2 + 3)(2x)}{(x^2 + 1)^2} = 0$	**M1**
$(0, 3)$ ((c) Total: 2)	**A1**
[Sketch: axes and correct asymptote labelled.]	**B1**
[Correct shape and position.]	**B1**
States $(0, 3)$ coordinates of intersection with axes, may be seen on diagram. ((d) Total: 3)	**B1**
[Sketch: FT from sketch in (d).]	**B1FT**
$\dfrac{x^2 + 1}{x^2 + 3} = \dfrac{1}{2}$	**M1**
$x^2 = 1 \Rightarrow x = \pm 1$	**A1**
$-1 < x < 1$ ((e) Total: 4)	**A1**
	Total: 15

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Q36

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Mark scheme.

$x = -\frac{1}{2}, \quad x = 4$	**B1**
Vertical asymptotes.	
$y = 1$	**B1**
Horizontal asymptote.	
	(2)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(2x^2 - 7x - 4)(4x - 5) - (2x^2 - 5x)(4x - 7)}{(2x^2 - 7x - 4)^2}$	**M1**
Finds $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.	
$4x^2 + 16x - 20 = 0$	**M1**
Sets equal to 0 and forms equation.	
$\left(-5, \frac{25}{27}\right) \quad \left(1, \frac{1}{3}\right)$	**A1A1**
	(4)
[Graph: correct shape, both branches, axes and asymptotes labelled.]	**B1**
Axes and asymptotes labelled.	
[Correct shape and position.]	**B1**
$(0, 0), \ \left(\frac{5}{2}, 0\right)$	**B1**
States coordinates of intersections with axes. May be seen on their diagram.	
	(3)
[Graph of $\lvert y \rvert$: correct shape and position.]	**B1**
	(1)
$\dfrac{2x^2 - 5x}{2x^2 - 7x - 4} = \dfrac{1}{9}$ or $\dfrac{2x^2 - 5x}{2x^2 - 7x - 4} = -\dfrac{1}{9}$	**M2**

$16x^2 - 38x + 4 = 0$ or $20x^2 - 52x - 4 = 0$	
Finds critical points, award M1 for each case.	
$x = \frac{19}{16} + \frac{3}{16}\sqrt{33}, \quad x = \frac{19}{16} - \frac{3}{16}\sqrt{33}$ or $x = \frac{13}{10} + \frac{3}{10}\sqrt{21}, \quad x = \frac{13}{10} - \frac{3}{10}\sqrt{21}$	**A1A1**
$\frac{13}{10} - \frac{3}{10}\sqrt{21} < x < \frac{19}{16} - \frac{3}{16}\sqrt{33}, \quad \frac{19}{16} + \frac{3}{16}\sqrt{33} < x < \frac{13}{10} + \frac{3}{10}\sqrt{21}$	**A1**
	(5)
	Total: 15

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Q37

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Mark scheme.

$x = -a$	**B1**
States vertical asymptote.	
$y = \dfrac{(x+a)(x-a)+a^2+a}{x+a} = x - a + \dfrac{a^2+a}{x+a}$	**M1**
Finds oblique asymptote.	
$y = x - a$	**A1**
	(3)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - (a^2+a)(x+a)^{-2} = 0$ or $x^2 + 2ax - a = 0$	**M1**
Differentiates	
Sets correct derivative to zero or ACF of quadratic equation	**A1**
$x = -a \pm \sqrt{a^2+a}$	**A1**
CAO. No equivalent forms.	
	(3)
[Graph: axes shown and asymptotes labelled.]	**B1FT**
FT on asymptotes from (a)	
[Lower branch correct.]	**B1**
[Upper branch with $(0, 1)$ stated or 1 marked on vertical axis and no others.]	**B1**
	(3)
[Graph of $\left\lvert \dfrac{x^2+a}{x+a} \right\rvert$: correct shape, FT from sketch in (c).]	**B1FT**
	(1)
$\dfrac{x^2+a}{x+a} = a \Rightarrow x^2 - ax + a - a^2 = 0$	**M1**

Finds critical points.	
Or $\dfrac{x^2+a}{x+a} = -a \Rightarrow x^2 + ax + a + a^2 = 0$	**B1**
Quadratic equation sufficient for B1. ISW. May state that $\dfrac{x^2+a}{x+a} = -a$ has no real solutions since $a > 0$.	
$a^2 - 4(a - a^2) = 5a^2 - 4a > 0$	**M1**
Uses that discriminant is positive to get a range of values for a. Cannot be $\geqslant 0$ so M0.	
$a > \frac{4}{5}$.	**A1**
	(4)
	Total: 14

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Q38

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Mark scheme.

$x = -\frac{3}{2} - \frac{1}{2}\sqrt{5}, \quad x = -\frac{3}{2} + \frac{1}{2}\sqrt{5}$	**B1**
Vertical asymptotes.	
$y = 0$	**B1**
Horizontal asymptote.	
	(2)
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2+3x+1)(1) - (x+2)(2x+3)}{(x^2+3x+1)^2}$	**M1**
Finds $\frac{\mathrm{d}y}{\mathrm{d}x}$.	
$x^2 + 4x + 5 = 0$	**M1**
Sets equal to 0 and forms quadratic equation.	
Discriminant is $4^2 - 20 = -4 < 0$ so no real roots.	**A1**
AG	
	(3)
[Sketch: axes and asymptotes labelled, correct shape and position, states coordinates of intersections with axes, $(0, 2)$ and $(-2, 0)$, may be seen on diagram.]	
Axes and asymptotes labelled.	**B1**
Correct shape and position.	**B1**
States coordinates of intersections with axes, $(0, 2)$ and $(-2, 0)$, may be seen on diagram.	**B1**
	(3)
[Sketch: reflection of negative parts of C in the x-axis. From a correct graph and with correct shape at $(-2, 0)$.]	**B1FTB1**
FT from sketch in part (c)	

From a correct graph and with correct shape at $(-2, 0)$.	
	(2)
$\dfrac{x+2}{x^2+3x+1} = 2$ or $\dfrac{x+2}{x^2+3x+1} = -2$	**M2**
$-2x^2 - 5x = 0$ or $2x^2 + 7x + 4 = 0$	
Finds critical points, award M1 for each case.	
$x = -\dfrac{5}{2},\ x = 0$ or $x = -\dfrac{7}{4} - \dfrac{1}{4}\sqrt{17},\ x = -\dfrac{7}{4} + \dfrac{1}{4}\sqrt{17}$	**A1**
$-\dfrac{7}{4} - \dfrac{1}{4}\sqrt{17} < x < -\dfrac{5}{2},\ -\dfrac{7}{4} + \dfrac{1}{4}\sqrt{17} < x < 0,\ x \neq -\dfrac{3}{2} \pm \dfrac{1}{2}\sqrt{5}$	**A1**
$-\dfrac{7}{4} - \dfrac{1}{4}\sqrt{17} < x < -\dfrac{3}{2} - \dfrac{1}{2}\sqrt{5},\ -\dfrac{3}{2} - \dfrac{1}{2}\sqrt{5} < x < -\dfrac{5}{2},\ -\dfrac{7}{4} + \dfrac{1}{4}\sqrt{17} < x < -\dfrac{3}{2} + \dfrac{1}{2}\sqrt{5},$ $-\dfrac{3}{2} + \dfrac{1}{2}\sqrt{5} < x < 0.$	
	(4)
	Total: 14

Q39

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Mark scheme.

$x = -1$	**B1**
Vertical asymptote.	
$y = \dfrac{x(x+1)+1}{x+1}$	**M1**
Oblique asymptote.	
$y = x$	**A1**
	(3)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2+2x}{(x+1)^2} = 0$	**M1**
Sets $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.	
$(0,1),\ (-2,-3)$	**A1A1**
1 mark for each correct coordinate. SCB1 if just the correct x values listed	
	(3)
[Sketch: axes and asymptotes labelled, left branch correct with all approaches to the asymptotes correct, right branch correct with a smooth minimum point.]	
Axes and asymptotes labelled.	**B1FT**
Left branch correct with all approaches to the asymptotes correct.	**B1**
Right branch correct with a smooth minimum point.	**B1**
	(3)
[Sketch: symmetrical about $x = 0$ and follow through from part (c) right branch; correct shape of a curve and smooth at $(0,1)$ from a correct graph in part (c).]	**B1FTB1**
Symmetrical about $x = 0$ and follow through from part (c) right branch	
Correct shape of a curve and smooth at $(0,1)$ from a correct graph in part (c)	
	(2)

$\dfrac{x^2+x+1}{x+1} = 3$	**M1**
$x^2 - 2x - 2 = 0$	
Finds critical point.	
$x = 1 + \sqrt{3}$	**A1**
For selecting just the positive value of x.	
$-1 - \sqrt{3} < x < 1 + \sqrt{3}$	**A1**
	(3)
	Total: 14

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Q40

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Mark scheme.

$x = 1.8$	**B1**
$y = \dfrac{(10x-18)\left(x+\frac{7}{10}\right)-\frac{27}{5}}{10x-18} = x + \dfrac{7}{10} - \dfrac{\frac{27}{5}}{10x-18}$	**M1**
Must be an attempt to find the constant term.	
$y = x + 0.7$	**A1**
	(3)
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{100x^2-360x+378}{(10x-18)^2}$	**M1**
Differentiates.	
$50x^2 - 180x + 189 = 0 \left(\text{or } \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 + \dfrac{54}{(10x-18)^2}\right)$	**A1**
Forms quadratic equation if using discriminant method. (Or writes $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in a form to show it is positive)	
$180^2 - 4(50)(189) = -5400 < 0$ (or $y' > 0$) $\Rightarrow$ No stationary points	**M1A1**
Consideration of discriminant or sign of y' with correct conclusion.	
	(4)
Axes and asymptotes labelled; branches correct.	**B1B1**
$(0,1)$, $(-0.9,0)$, $(2,0)$	**B1**
	(3)
FT from their attempt in (c).	**B1**
Everything correct with sharp changes of direction at intersections with x axis.	**B1**
	(2)
$\dfrac{10x^2-11x-18}{10x-18} = -4 \Rightarrow x = \dfrac{-29 \pm \sqrt{4441}}{20}$	
$\dfrac{10x^2-11x-18}{10x-18} = 4$	**M1**

Forms an equation to find required critical values for x.	
$10x^2 - 51x + 54 = 0$	**A1**
$x = \frac{3}{2}$, $x = \frac{18}{5}$	**A1**
$p = \frac{3}{2}$, $q = \frac{18}{5}$, clearly identified in correct order.	**A1**
	(4)

Total: 16

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Written by Ali Hashir.

Questions available in the companion questions booklet.

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