Further Math with Ali Hashir

Proof by Induction

Topical Questions (CIE 9231)

Compiled from CIE 9231 Further Mathematics Paper 1 past papers (2015–2025), sorted by topic. Each question is numbered for this booklet, with its original paper code and session shown beneath.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Q1

9231/11, 9231/12 • May/June 2015 • Question 3

The sequence $a_1, a_2, a_3, \ldots$ is such that $a_1 > 5$ and $a_{n+1} = \frac{4a_n}{5} + \frac{5}{a_n}$ for every positive integer n. Prove by mathematical induction that $a_n > 5$ for every positive integer n. [5]

Prove also that $a_n > a_{n+1}$ for every positive integer n. [2]

Q2

9231/13 • May/June 2015 • Question 3

Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1} = \frac{n}{2n+1}.$$

[6]

State the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

Q3

9231/11, 9231/12, 9231/13 • October/November 2015 • Question 3

Given that a is a constant, prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x\mathrm{e}^{ax}) = na^{n-1}\mathrm{e}^{ax} + a^n x\mathrm{e}^{ax}.$$

[6]

Q4

9231/11, 9231/12 • May/June 2016 • Question 3

Prove by mathematical induction that, for all positive integers n, $10^n + 3 \times 4^{n+2} + 5$ is divisible by 9. [6]

Q5

9231/13 • May/June 2016 • Question 2

It is given that a diagonal of a polygon is a line joining two non-adjacent vertices. Prove, by mathematical induction, that an n-sided polygon has $\frac{1}{2}n(n-3)$ diagonals, where $n \geq 3$. [6]

Q6

9231/11, 9231/12, 9231/13 • October/November 2016 • Question 4

Using factorials, show that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. [2]

Hence prove by mathematical induction that

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n}x^n$$

for every positive integer n. [4]

Q7

9231/11, 9231/12 • May/June 2017 • Question 2

Prove, by mathematical induction, that $5^n + 3$ is divisible by 4 for all non-negative integers n. [5]

Q8

9231/13 • May/June 2017 • Question 3

Prove, by mathematical induction, that $\sum_{r=1}^{n} r \ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(n+1)^n}{n!}\right)$ for all positive integers n. [6]

Q9

9231/11, 9231/12, 9231/13 • October/November 2017 • Question 3

(i) Show that

$$\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(x^{n+1}\ln x\right) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n + (n+1)x^n \ln x\right).$$

[2]

(ii) Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n \ln x) = n!\left(\ln x + 1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).$$

[5]

Q10

9231/11, 9231/12 • May/June 2018 • Question 2

It is given that $f(n) = 2^{3n} + 8^{n-1}$. By simplifying $f(k) + f(k+1)$, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 9 for every positive integer n. [6]

Q11

9231/13 • May/June 2018 • Question 9

For the sequence u_1, u_2, u_3, ..., it is given that $u_1 = 8$ and

$$u_{r+1} = \frac{5u_r - 3}{4}$$

for all r.

(i) Prove by mathematical induction that

$$u_n = 4\left(\tfrac{5}{4}\right)^n + 3,$$

for all positive integers n. [5]

(ii) Deduce the set of values of x for which the infinite series

$$(u_1 - 3)x + (u_2 - 3)x^2 + \ldots + (u_r - 3)x^r + \ldots$$

is convergent. [2]

(iii) Use the result given in part (i) to find surds a and b such that

$$\sum_{n=1}^{N} \ln(u_n - 3) = N^2 \ln a + N \ln b.$$

[3]

Q12

9231/11, 9231/13 • October/November 2018 • Question 3

The sequence of positive numbers u_1, u_2, u_3, ... is such that $u_1 < 3$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{4u_n + 9}{u_n + 4}.$$

(i) By considering $3 - u_{n+1}$, or otherwise, prove by mathematical induction that $u_n < 3$ for all positive integers n. [5]

(ii) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

Q13

9231/12 • October/November 2018 • Question 6

It is given that $y = \mathrm{e}^x u$, where u is a function of x. The rth derivatives $\dfrac{\mathrm{d}^r y}{\mathrm{d}x^r}$ and $\dfrac{\mathrm{d}^r u}{\mathrm{d}x^r}$ are denoted by $y^{(r)}$ and $u^{(r)}$ respectively. Prove by mathematical induction that, for all positive integers n,

$$y^{(n)} = \mathrm{e}^x \left(\binom{n}{0} u + \binom{n}{1} u^{(1)} + \binom{n}{2} u^{(2)} + \ldots + \binom{n}{r} u^{(r)} + \ldots + \binom{n}{n} u^{(n)} \right).$$

[8]

[You may use without proof the result $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.]

Q14

9231/11, 9231/12 • May/June 2019 • Question 8

(i) Prove by mathematical induction that, for $z \neq 1$ and all positive integers n,

$$1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1}.$$

[5]

(ii) By letting $z = \frac{1}{2}(\cos\theta + \mathrm{i}\sin\theta)$, use de Moivre's theorem to deduce that

$$\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m \sin m\theta = \frac{2\sin\theta}{5 - 4\cos\theta}.$$

[5]

Q15

9231/13 • May/June 2019 • Question 1

Prove by mathematical induction that $3^{3n} - 1$ is divisible by 13 for every positive integer n.
[5]

Q16

9231/11, 9231/12, 9231/13 • October/November 2019 • Question 2

It is given that $y = \ln(ax + 1)$, where a is a positive constant. Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+1)^n}.$$

[6]

Q17

9231/13 • May/June 2020 • Question 2

The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 2^n - 1$ for all positive integers n. [5]

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

Q18

9231/11, 9231/13 • October/November 2020 • Question 5

Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\sin x) = (-1)^{n-1}\big(x\cos x + (2n-1)\sin x\big).$$

[7]

Q19

9231/12 • October/November 2020 • Question 2

Prove by mathematical induction that $7^{2n} - 1$ is divisible by 12 for every positive integer n. [5]

Q20

9231/11, 9231/12 • May/June 2021 • Question 1

Prove by mathematical induction that $2^{4n} + 31^n - 2$ is divisible by 15 for all positive integers n. [6]

Q21

9231/13 • May/June 2021 • Question 3

(a) Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n} \left(5r^4 + r^2\right) = \tfrac{1}{2}n^2(n+1)^2(2n+1).$$

[6]

(b) Use the result given in part (a) together with the List of formulae (MF19) to find $\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]

Q22

9231/11, 9231/13 • October/November 2021 • Question 3

The sequence of real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^3.$$

(a) Prove by mathematical induction that $\ln a_n \geqslant 3^{n-1} \ln 2$ for all integers $n \geqslant 2$. [6]

[You may use the fact that $\ln\left(x + \frac{1}{x}\right) > \ln x$ for $x > 0$.]

(b) Show that $\ln a_{n+1} - \ln a_n > 3^{n-1} \ln 4$ for $n \geqslant 2$. [2]

Q23

9231/12 • October/November 2021 • Question 2

It is given that $y = xe^{ax}$, where a is a constant.

Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n y}{dx^n} = \left(a^n x + na^{n-1}\right) e^{ax}.$$

[6]

Q24

9231/11, 9231/12 • May/June 2022 • Question 3

The sequence of positive numbers u_1, u_2, u_3, ... is such that $u_1 > 4$ and, for $n \geq 1$,

$$u_{n+1} = \frac{u_n^2 + u_n + 12}{2u_n}.$$

(a) By considering $u_{n+1} - 4$, or otherwise, prove by mathematical induction that $u_n > 4$ for all positive integers n. [5]

(b) Show that $u_{n+1} < u_n$ for $n \geq 1$. [3]

Q25

9231/11, 9231/13 • October/November 2022 • Question 2

Prove by mathematical induction that, for all positive integers n, $7^{2n} + 97^n - 50$ is divisible by 48. [6]

Q26

9231/12 • October/November 2022 • Question 4

The function f is such that $\mathrm{f}''(x) = \mathrm{f}(x)$.

Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}\big(x\,\mathrm{f}(x)\big) = x\,\mathrm{f}'(x) + (2n-1)\,\mathrm{f}(x).$$

[7]

Q27

9231/11, 9231/12 • May/June 2023 • Question 1

Let $\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) Prove by mathematical induction that, for all positive integers n,

$$2\mathbf{A}^n = \begin{pmatrix} 2 \times 3^n & 0 \\ 3^n - 1 & 2 \end{pmatrix}.$$

[5]

(b) Find, in terms of n, the inverse of $\mathbf{A}^n$. [2]

Q28

9231/13 • May/June 2023 • Question 1

Prove by mathematical induction that, for all positive integers n, $5^{3n} + 32^n - 33$ is divisible by 31. [6]

Q29

9231/11, 9231/13 • October/November 2023 • Question 2

Prove by mathematical induction that, for all positive integers n,

$$1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

[6]

Q30

9231/12 • October/November 2023 • Question 2

Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2 \mathrm{e}^x\right) = \left(x^2 + 2nx + n(n-1)\right)\mathrm{e}^x.$$

[6]

Q31

9231/11, 9231/12 • May/June 2024 • Question 2

Prove by mathematical induction that $6^{4n} + 38^n - 2$ is divisible by 74 for all positive integers n. [6]

Q32

9231/12 • October/November 2024 • Question 1

The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 4$ and $u_{n+1} = 3u_n - 2$ for $n \geq 1$.

Prove by induction that $u_n = 3^n + 1$ for all positive integers n. [5]

Q33

9231/11, 9231/13 • October/November 2024 • Question 2

Prove by mathematical induction that, for all positive integers n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\tan^{-1} x\right) = P_n(x)\left(1+x^2\right)^{-n},$$

where $P_n(x)$ is a polynomial of degree $n-1$. [6]

Q34

9231/11, 9231/12 • May/June 2025 • Question 3

The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 5$ and $u_{n+1} = 6u_n + 5$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 6^n - 1$ for all positive integers n. [5]

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

Q35

9231/13 • May/June 2025 • Question 2

Prove by mathematical induction that $2025^n + 47^n - 2$ is divisible by 46 for all positive integers n. [6]

Q36

9231/11, 9231/13 • October/November 2025 • Question 3

Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\cos x) = (-1)^n\,(x\sin x - (2n-1)\cos x)\,.$$

[7]

Q37

9231/12 • October/November 2025 • Question 3

The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 5$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{6u_n + 5}{u_n + 2}.$$

(a) By considering $5 - u_{n+1}$, prove by mathematical induction that $u_n < 5$ for all positive integers n. [5]

(b) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

Q38

9231/14 • October/November 2025 • Question 1

Prove by mathematical induction that $\left(\frac{6}{5}\right)^n \geqslant 1 + \frac{1}{5}n$ for all positive integers n. [5]

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Written by Ali Hashir.

Answers available in the companion mark scheme booklet.