Further Math with Ali Hashir

Proof by Induction

Topical Mark Schemes (CIE 9231)

Mark schemes corresponding to the topical questions booklet for this topic, compiled from official CIE 9231 Paper 1 marking schemes (2015–2025). Question numbers match the questions booklet.

About this resource: I teach CIE A Level Further Mathematics, and over the years I found that there simply weren't many well-organised, topic-wise resources available for this subject – it's a niche A Level with relatively few candidates, so it doesn't get much attention from resource makers. These topical booklets are my attempt to fix that: every past-paper question, sorted by topic, with its official mark scheme alongside it.

Help us improve these documents for future students. If you spot a mistake, email us at **vectorspacepk@gmail.com**.

What kind of mistakes might you find? These booklets are compiled by hand from hundreds of past-paper PDFs. I've checked them as carefully as I can, but I'm only human, and the odd error may have slipped through – so please help me catch what I missed. Things to look out for:

- Spelling or typing mistakes
- Wrong final answers
- A mark scheme that doesn't line up with its question (misaligned pages)
- A mark scheme that belongs to the wrong question entirely
- Pre-2020 questions that are technically out of the current syllabus (the syllabus changed in 2020, but some older questions are still included here for practice)
- Missing, extra, or wrongly-labelled marks (e.g. an M1 or A1 that shouldn't be there, or one that's missing)
- Diagrams or images that are unclear, missing, or don't match the question
- Questions placed under the wrong topic

If you spot any of these (or anything else that looks off), please let me know.

This is **Version 1** of this booklet. As readers point out errors, I'll keep correcting and improving these documents, so I'd recommend checking back on the website from time to time and using the latest version rather than relying on a saved copy.

If you think these documents brought you value, consider following me on social media – it helps more than you'd think.

Contents

Click a question number below to jump straight to its mark scheme. Every box is a live link.

For the best hyperlink experience, open this PDF in a dedicated reader – such as Adobe Acrobat, Apple Preview, or your browser's built-in PDF viewer – rather than a basic in-app or embedded preview, where links sometimes don't work.

Q1

9231/11, 9231/12 • May/June 2015 • Question 3

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Mark scheme.

$a_1 > 5$ (given) $\Rightarrow \mathrm{H}_1$ is true.	**B1**
Assume H_k is true for some positive integer k, i.e. $a_k = 5 + \delta$, where $\delta > 0$.	**B1**
$a_{k+1} - 5 = \dfrac{4a_k^2 + 25}{5a_k} - 5 = \dfrac{4a_k^2 + 25 - 25a_k}{5a_k} = \dfrac{(4a_k - 5)(a_k - 5)}{5a_k} > 0, \Rightarrow a_{k+1} > 5$	**M1A1**
Or	
$a_{k+1} = \dfrac{4}{5}(5+\delta) + \dfrac{5}{5+\delta}, \ = 4 + \dfrac{4}{5}\delta + (1 - \dfrac{\delta}{5} + \dfrac{\delta^2}{25} - \dots)$ for $0 < \delta < 5$	**(M1)(A1)**
$= 5 + \dfrac{3}{5}\delta + 0(\delta^2) \geqslant a_{k+1} > 5$, ($\delta \geqslant 5$ is trivial).	
$\mathrm{H}_k \Rightarrow \mathrm{H}_{k+1}$ and H_1 is true, hence by mathematical induction, the result is true for all $n \in \mathbb{Z}^+$ (N.B. The minimum requirement is 'true for all positive integers'.)	**A1**
	(5)
$a_{k+1} - a_k = \dfrac{5}{a_k} - \dfrac{1}{5}a_k$	**M1**
$\dfrac{5}{a_k} < 1$ and $\dfrac{1}{5}a_k > 1 \Rightarrow a_{k+1} - a_k < 0 \Rightarrow a_{k+1} < a_k$	**A1**
	(2)
	Total: 7

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Q2

9231/13 • May/June 2015 • Question 3

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Mark scheme.

$\mathrm{H}_k : \sum_{r=1}^{k} \frac{1}{(2r)^2 - 1} = \frac{k}{2k+1}$ is true for some integer k.	**B1**
$\frac{1}{2^2 - 1} = \frac{1}{3} = \frac{1}{2 \times 1 + 1} \Rightarrow \mathrm{H}_1$ is true.	**B1**
$\frac{k}{2k+1} + \frac{1}{(2k+2)^2 - 1} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)}$	**M1A1**
$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2[k+1]+1}$	**A1**
$\therefore \mathrm{H}_k \Rightarrow \mathrm{H}_{k+1}$	
$\therefore$ (By Principle of Mathematical Induction) H_n is true for all positive integers n. (This mark requires all previous marks.)	**A1**
	(6)
$\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1} = \frac{1}{2}$	**B1**
	(1)
	Total: 7

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Q3

9231/11, 9231/12, 9231/13 • October/November 2015 • Question 3

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Mark scheme.

$n = 1$ in formula gives $a^0\mathrm{e}^{ax} + ax\mathrm{e}^{ax} = \mathrm{e}^{ax} + ax\mathrm{e}^{ax}$	**B1**
$\dfrac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^{ax}) = \mathrm{e}^{ax} \times 1 + x.a\mathrm{e}^{ax} = \mathrm{e}^{ax} + ax\mathrm{e}^{ax} \Rightarrow \mathrm{H}_1$ is true oe	**B1**
Assume H_k is true, i.e. $\dfrac{\mathrm{d}^k}{\mathrm{d}x^k}(x\mathrm{e}^{ax}) = ka^{k-1}\mathrm{e}^{ax} + a^k x\mathrm{e}^{ax}$.	**B1**
$\dfrac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}(x\mathrm{e}^{ax}) = ka^k\mathrm{e}^{ax} + a^k\mathrm{e}^{ax} + a^{k+1}x\mathrm{e}^{ax}$	**M1**
$= (k+1)a^k\mathrm{e}^{ax} + a^{k+1}x\mathrm{e}^{ax}$	**A1**
$\Rightarrow \mathrm{H}_{k+1}$ is true, hence by PMI H_n is true for all positive integers n.	**A1**
	Total: 6

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Q4

9231/11, 9231/12 • May/June 2016 • Question 3

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Mark scheme.

For $n = 1$ $10 + 192 + 5 = 207 = 9 \times 23 \Rightarrow H_1$ is true.	**B1**
Assume H_k is true for some positive integer $k \Rightarrow 10^n + 3.4^{n+2} + 5 = 9\alpha$	**B1**
Let $f(n) = 10^n + 3.4^{n+2} + 5$	
Hence $f(n+1) - f(n) = 10^n(10-1) + 3.4^{n+2}(4-1)$	**M1**
$= 9(10^n + 4^{n+2})$	
$= 9\beta$	**A1**
Hence $f(n+1)(= 9(\beta + \alpha)) \Rightarrow H_{k+1}$ is true	**A1**
H_1 is true and $H_k \Rightarrow H_{k+1}$, hence by PMI H_n is true for all positive integers n.	**A1**
	(6)
N.B. Or can show $f(n+1) = 9(10\alpha - 2.4^{n+2} - 5)$ for M1A1A1. (3rd, 4th & 5th marks)	
	Total: 6

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Q5

9231/13 • May/June 2016 • Question 2

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Mark scheme.

With $n = 3$, $\frac{1}{2}n(n-3) = 0$	**M1**
A triangle has no diagonals $\Rightarrow H_3$ is true.	**A1**
Assume H_k is true: A k-gon has $\frac{1}{2}k(k-3)$ diagonals for some integer ≥ 3	**B1**
Adding an extra vertex, a further $(k-1)$ diagonals can be drawn.	**M1**
$\frac{1}{2}k(k-3) + k - 1 = \frac{k^2 - 3k + 2k - 2}{2} = \frac{(k+1)(k-2)}{2}$	**A1**
$= \frac{1}{2}(k+1)(k+1-3)$ (So $H_k \Rightarrow H_{k+1}$)	**A1**
$\Rightarrow H_n$ is true for all integers $n \geq 3$.	
	(6)
	Total: 6

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Q6

9231/11, 9231/12, 9231/13 • October/November 2016 • Question 4

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Mark scheme.

$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r)!}\left(\frac{1}{n-r+1} + \frac{1}{r}\right)$	**M1**
$= \frac{n!}{(r-1)!(n-r)!}\left(\frac{r+n-r+1}{r(n-r+1)}\right) = \frac{(n+1)!}{r!(n-r+1)!} = \binom{n+1}{r}$	**A1**
	(2)
$(a+x)^1 = \binom{1}{0}a + \binom{1}{1}x = a + x \Rightarrow \mathrm{H}_1$ is true.	**B1**
Assume H_k is true, i.e. $(a+x)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}x + \ldots + \binom{k}{r}a^{k-r}x^r + \ldots + \binom{k}{k}x^k$	**B1**
Multiplying by $(a+x)$, the coefficient of $a^{k-r+1}x^r$ is: $\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r} \Rightarrow \mathrm{H}_{k+1}$ is true.	**M1**
Hence H_n is true for all positive integers.	**A1**
	(4)
	Total: 6

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Q7

9231/11, 9231/12 • May/June 2017 • Question 2

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Mark scheme.

Let P_n be the proposition that $5^n + 3$ is divisible by 4; $5^0 + 3 = 4 \Rightarrow P_0$ is true (allow P_1) (Some explanation of what P_k being true means)	**B1**
Assume that P_k is true for some non-negative integer k. (or e.g. $5^k + 3 = 4\alpha$ for 2nd **B1**)	**B1**
$5^{k+1} + 3 = 5(4\alpha - 3) + 3$ (Alt method: Use f(k+1) − f(k) **M1 A1**)	**M1**
$= 20\alpha - 12 = 4(5\alpha - 3)$ (or shows that $5^{k+1} + 3 = 5.5^k + 5.3 - 4.3 = 5(5^k + 3) - 4.3$)	**A1**
P_0 is true and $P_k \Rightarrow P_{k+1}$, hence P_n is true for all non-negative integers n.	**A1**
	Total: 5

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Q8

9231/13 • May/June 2017 • Question 3

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Mark scheme.

When $n = 1$ $\quad 1 \times \ln 2 = \ln 2 = \ln\left(\frac{2^1}{1!}\right) \Rightarrow (\mathrm{H}_1 \text{ is true})$	**B1**
Assume, for some positive integer k, that $\sum_{r=1}^{k} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{[k+1]^k}{k!}\right)$	**B1**
Hence $\sum_{r=1}^{k+1} r\ln\left(\frac{r+1}{r}\right) = \ln\left(\frac{(k+1)^k}{k!}\right) + (k+1)\ln\left(\frac{k+2}{k+1}\right)$	**B1**
$= \ln\frac{(k+1)^k(k+2)^{k+1}}{k!(k+1)^{k+1}}$	**M1**
$= \ln\frac{(k+1)^k(k+2)^{k+1}}{(k+1)!(k+1)^k} = \ln\frac{(k+2)^{k+1}}{(k+1)!}$	**A1**
Thus $H_k \Rightarrow H_{k+1}$ and hence by PMI H_n is true for all positive integers.	**A1**
	Total: 6

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Q9

9231/11, 9231/12, 9231/13 • October/November 2017 • Question 3

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Mark scheme.

$\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}\left(x^{n+1}\ln x\right) = \frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left(x^{n+1}\cdot\frac{1}{x} + (n+1)x^{n}\ln x\right) =$	**M1A1**
$\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\left(x^{n} + (n+1)x^{n}\ln x\right)$	**AG**
	(2)
Assume H_k is true $\Rightarrow \frac{\mathrm{d}^{k}}{\mathrm{d}x^{k}}(x^{k}\ln x) = k!\left\{\ln x + 1 + \frac{1}{2} + \ldots + \frac{1}{k}\right\}$ (Statement of H_k seen)	**B1**
$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^{k+1}\ln x\right) = \frac{\mathrm{d}^{k}}{\mathrm{d}x^{k}}\left(x^{k} + [k+1]x^{k}\ln x\right)$	**M1**
$= k! + [k+1]k!\left\{\ln x + 1 + \frac{1}{2} + \ldots + \frac{1}{k}\right\}$	**A1**
$= (k+1)!\left\{\ln x + 1 + \frac{1}{2} + \ldots + \frac{1}{k+1}\right\} \Rightarrow H_{k+1}$ is true	**A1**
Check H_1 is true **and** H_k is true $\Rightarrow H_{k+1}$ is true; hence, by PMI, H_n is true for all positive integers n.	**A1**
	(5)
	Total: 7

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Q10

9231/11, 9231/12 • May/June 2018 • Question 2

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Mark scheme.

We have that $f(1) = 9$ is divisible by 9.	**B1**
Assume that $f(k)$ is divisible by 9.	**B1**
$f(k+1) + f(k) = 2^{3k+3} + 8^k + 2^{3k} + 8^{k-1} =$	**M1**
$2^3.2^{3k} + 8.8^{k-1} + 2^{3k} + 8^{k-1}$ OE	**A1**
$= 9\left(2^{3k} + 8^{k-1}\right)$ OE so $f(k+1)$ is divisible by 9.	**A1**
So if $f(\text{k})$ is divisible by 9, so is $f(k+1)$, (and $f(1)$ is divisible by 9), $f(n)$ is divisible by 9 for every integer $n \geqslant 1$	**A1**
	Total: 6

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Q11

9231/13 • May/June 2018 • Question 9

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Mark scheme.

P_n: $u_n = 4\left(\frac{5}{4}\right)^n + 3$	**B1**
Let $n = 1$ then $4\left(\frac{5}{4}\right) + 3 = 8 \Rightarrow P_1$ true.	
Assume P_k is true for some k. Then	**B1**
$u_{k+1} = \frac{1}{4}\left(5\left(4\left(\frac{5}{4}\right)^k + 3\right) - 3\right) = \textit{correct step}$	**M1**
$= 4\left(\frac{5}{4}\right)^{k+1} + 3,$	**A1**
So $\mathrm{P}_k \Rightarrow \mathrm{P}_{k+1}$. Therefore, by induction, P_n is true for all positive integers.	**A1**
	(5)
$(u_n - 3)x^n = 4x^n\left(\frac{5}{4}\right)^n = 4\left(\frac{5x}{4}\right)^n$ so $r = \left(\frac{5x}{4}\right)$	**M1**
So series is convergent for $-1 < \frac{5x}{4} < 1 \Rightarrow -\frac{4}{5} < x < \frac{4}{5}$	**A1**
	(2)
$\sum_{n=1}^{N} \ln(u_n - 3) = \sum_{n=1}^{N} \ln\left(4\left(\frac{5}{4}\right)^n\right)$	**M1**
$= \left(\ln\frac{5}{4}\right)\sum_{n=1}^{N} n + \sum_{n=1}^{N} \ln 4$	
Alt method: $\sum_{n=1}^{N} \ln(u_n - 3) = \ln \prod 4\left(\frac{5}{4}\right)^n = \ln 4^N \prod_{1}^{N}\left(\frac{5}{4}\right)^n = N\ln 4 + \ln\left(\frac{5}{4}\right)^{\sum n}$	**M1**
$= \frac{1}{2}N(N+1)\ln\frac{5}{4} + N\ln 4$ Use $\sum_{n=1}^{N} n = \frac{1}{2}N(N+1)$.	**M1**
$= N^2\ln\frac{\sqrt{5}}{2} + N\ln(2\sqrt{5}) \Rightarrow a = \frac{\sqrt{5}}{2},\ b = 2\sqrt{5}$ oe	**A1**
Alt method: Writes series as an AP M1, uses summation formula M1 Correct answer A1	
	(3)
	Total: 10

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Q12

9231/11, 9231/13 • October/November 2018 • Question 3

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Mark scheme.

$u_1 < 3$ (given) (States base case.)	**B1**
Assume that $u_k < 3$ (States inductive hypothesis.)	**B1**
Then $3 - u_{k+1} = 3 - \dfrac{4u_k + 9}{u_k + 4} = \dfrac{-u_k + 3}{u_k + 4} > 0 \Rightarrow u_{k+1} < 3$	**M1A1**
Hence, by induction, $u_n < 3$ for all $n \geqslant 1$. (States conclusion.)	**B1**
	(5)
$u_{n+1} - u_n = \dfrac{4u_n + 9}{u_n + 4} - u_n = \dfrac{-u_n^2 + 9}{u_n + 4}$ (Considers $u_{n+1} - u_n$.)	**M1A1**
So $u_n < 3 \Rightarrow u_{n+1} - u_n > 0$. (Uses $u_n < 3$)	**B1**
	(3)
	Total: 8

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Q13

9231/12 • October/November 2018 • Question 6

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Mark scheme.

$y^{(1)} = \mathrm{e}^x u^{(1)} + u\mathrm{e}^x = \mathrm{e}^x \left(\binom{1}{0}u + \binom{1}{1}u^{(1)}\right) \Rightarrow H_1$ is true (Shows base case using product rule)	**M1A1**
Assume that $H_k : y^{(k)} = \mathrm{e}^x \left(\binom{k}{0}u + \binom{k}{1}u^{(1)} + \dots + \binom{k}{r}u^{(r)} + \dots + \binom{k}{k}u^{(k)}\right)$ (States inductive hypothesis.)	**B1**
Then $y^{(k+1)} = \mathrm{e}^x \left(\binom{k}{0}u + \binom{k}{1}u^{(1)} + \dots + \binom{k}{r}u^{(r)} + \dots + \binom{k}{k}u^{(k)}\right) +$ (Differentiates using product rule)	**M1**
$\mathrm{e}^x \left(\binom{k}{0}u^{(1)} + \binom{k}{1}u^{(2)} + \dots + \binom{k}{r}u^{(r+1)} + \dots + \binom{k}{k}u^{(k+1)}\right)$	**M1A1**
$= \mathrm{e}^x \left(\binom{k}{0}u + \dots \left(\binom{k}{r} + \binom{k}{r-1}\right) u^r + \dots \binom{k}{k}u^{(k+1)}\right)$ (Shows application of $\binom{k}{r} + \binom{k}{r-1} = \binom{k+1}{r}$.)	
$= \mathrm{e}^x \left(\binom{k+1}{0}u + \dots \binom{k+1}{r}u^r + \dots \binom{k+1}{k+1}u^{(k+1)}\right)$ (Shows reasoning for first and last term correctly)	**B1**
So H_k implies H_{k+1} so, by induction, H_n is true for all $n \geqslant 1$. (States conclusion.)	**A1**
	(8)
	Total: 8

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Q14

9231/11, 9231/12 • May/June 2019 • Question 8

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Mark scheme.

$1 = \dfrac{z^1 - 1}{z - 1}$ so true when $n = 1$.	**B1**
Assume that $1 + z + \ldots + z^{k-1} = \dfrac{z^k - 1}{z - 1}$	**B1**
Then $1 + z + \ldots + z^{k-1} + z^k = \dfrac{z^k - 1}{z - 1} + z^k = \dfrac{z^k - 1 + z^k(z-1)}{z-1} = \dfrac{z^{k+1} - 1}{z - 1}$, so true when $n = k + 1$	**M1A1**
$H_k \to H_{k+1}$. Hence, by induction, true for all positive integers.	**A1**
	(5)
Since $\lvert z \rvert < 1$, $\displaystyle\sum_{m=0}^{\infty} z^m = \frac{-1}{z-1}$	**B1**
$\displaystyle\sum_{m=1}^{\infty} 2^{-m} \sin m\theta = \operatorname{Im}\left(\sum_{m=0}^{\infty} z^m\right) = \operatorname{Im}\left(\frac{-1}{\frac{1}{2}\cos\theta + \mathrm{i}\frac{1}{2}\sin\theta - 1}\right)$	**M1A1**
$\operatorname{Im}\left(\dfrac{-\left(\frac{1}{2}\cos\theta - 1 - \mathrm{i}\frac{1}{2}\sin\theta\right)}{\frac{1}{4}\cos^2\theta - \cos\theta + 1 + \frac{1}{4}\sin^2\theta}\right)$	**M1**
$\dfrac{\frac{1}{2}\sin\theta}{\frac{5}{4} - \cos\theta} = \dfrac{2\sin\theta}{5 - 4\cos\theta}$	**A1**
	(5)
	Total: 10

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Q15

9231/13 • May/June 2019 • Question 1

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Mark scheme.

$3^3 - 1 = 26$ is divisible by 13	**B1**
Assume that $3^{3k} - 1$ is divisible by 13 for some positive integer k	**B1**
Then $3^{3k+3} - 1 = 3^3 3^{3k} - 1 = 26 \cdot 3^{3k} + 3^{3k} - 1$	**M1**
is divisible by 13	**A1**
$H_k \Rightarrow H_{k+1}$. By induction, $3^{3n} - 1$ is divisible by 13 for every positive integer n.	**A1**
	Total: 5

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Q16

9231/11, 9231/12, 9231/13 • October/November 2019 • Question 2

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Mark scheme.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{a}{ax+1} = (-1)^0 \dfrac{0!\,a^1}{(ax+1)^1}$ so true for $n = 1$.	**M1A1**
Proves base case.	
Assume that $\dfrac{\mathrm{d}^k y}{\mathrm{d}x^k} = (-1)^{k-1} \dfrac{(k-1)!\,a^k}{(ax+1)^k}$ for some positive integer k.	**B1**
States inductive hypothesis.	
Then $\dfrac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = -ka(-1)^{k-1} \dfrac{(k-1)!\,a^k}{(ax+1)^{k+1}} = (-1)^k \dfrac{k!\,a^{k+1}}{(ax+1)^{k+1}}$ so true for $n = k+1$.	**M1A1**
Differentiates k^{th} derivative.	
By induction, true for every positive integer n.	**A1**
States conclusion.	
	Total: 6

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Q17

9231/13 • May/June 2020 • Question 2

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Mark scheme.

$u_1 = 1 = 2^1 - 1$	**B1**
Assume that it is true for $n = k$, so $u_k = 2^k - 1$.	**B1**
Then $u_{k+1} = 2(2^k - 1) + 1 = 2^{k+1} - 1$	**M1A1**
So, it is also true for $n = k + 1$. Hence, by induction, true for all positive integers.	**A1**
	(5)
$\dfrac{u_{2n}}{u_n} = \dfrac{2^{2n} - 1}{2^n - 1} = \dfrac{(2^n - 1)(2^n + 1)}{(2^n - 1)} = 2^n + 1$	**M1A1**
	(2)
	Total: 7

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Q18

9231/11, 9231/13 • October/November 2020 • Question 5

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Mark scheme.

$\dfrac{\mathrm{d}}{\mathrm{d}x}(x\sin x) = x\cos x + \sin x = (-1)^0(x\cos x + (2(1)-1)\sin x)$	**B1**
Checks base case using product rule.	
Assume true for $n = k$, so $\dfrac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}}(x\sin x) = (-1)^{k-1}(x\cos x + (2k-1)\sin x)$	**B1**
States inductive hypothesis.	
Then $\dfrac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}(x\sin x) = (-1)^{k-1}(-x\sin x + 2k\cos x)$	**M1A1**
Differentiates once. Must have correct LHS for A1.	
$\dfrac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}(x\sin x) = (-1)^{k-1}(-x\cos x - \sin x - 2k\sin x)$	**M1A1**
$= (-1)^k(x\cos x + (2k+1)\sin x)$	
Differentiates again.	
So, it is also true for $n = k+1$. Hence, by induction, true for all positive integers.	**A1**
States conclusion.	
	Total: 7

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Q19

9231/12 • October/November 2020 • Question 2

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Mark scheme.

$7^2 - 1 = 48$ is divisible by 12.	**B1**
Checks base case.	
Assume that $7^{2k} - 1$ is divisible by 12 for some positive integer k.	**B1**
States inductive hypothesis.	
Then $7^{2k+2} - 1 = 49 \cdot 7^{2k} - 1 = 48 \cdot 7^{2k} + 7^{2k} - 1$	**M1**
Or $(7^{2k+2} - 1) - (7^{2k} - 1) = 48 \cdot 7^{2k}$	
Separates $7^{2k} - 1$.	
$7^{2k+2} - 1$ is divisible by 12.	**A1**
So if is true for $n = k$ (could be written earlier) it is also true for $n = k + 1$. Hence, by induction, true for every positive integer n.	**A1**
Depends on previous M1 and A1.	
	Total: 5

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Q20

9231/11, 9231/12 • May/June 2021 • Question 1

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Mark scheme.

$2^4 + 31 - 2 = 45$ is divisible by 15	**B1**
Assume that $2^{4k} + 31^k - 2$ is divisible by 15 for some positive integer k.	**B1**
Then $2^{4k+4} + 31^{k+1} - 2 = (15+1)2^{4k} + (30+1)31^k - 2$	**M1 A1**
is divisible by 15 because $15 \times 2^{4k} + 30 \times 31^k$ is divisible by 15.	**A1**
Hence, by induction, true for every positive integer n.	**A1**
	Total: 6

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Q21

9231/13 • May/June 2021 • Question 3

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Mark scheme.

$5 \times 1^4 + 1^2 = \frac{1}{2}(2)^2(2+1)$ $[= 6]$ so H_1 is true.	**B1**
Assume that $\sum_{r=1}^{k}(5r^4 + r^2) = \frac{1}{2}k^2(k+1)^2(2k+1)$ for some k. (States inductive hypothesis including the algebraic form. If says for ALL k, then B0.)	**B1**
$\sum_{r=1}^{k+1}(5r^4 + r^2) = \frac{1}{2}k^2(k+1)^2(2k+1) + 5(k+1)^4 + (k+1)^2$	**M1**
$\frac{1}{2}(k+1)^2\left(2k^3 + k^2 + 10(k+1)^2 + 2\right)$ (Take out the factor of $(k+1)^2$ OR expands the summation expression and the target expression for $k+1$ and collects like terms for both.)	**M1**
$\frac{1}{2}(k+1)^2(2k^3+11k^2+20k+12) = \frac{1}{2}(k+1)^2(k+2)^2(2k+3)$ (Factorises or having expanded, checks explicitly. At least one intermediate step seen following the award of M1 before reaching the answer.)	**A1**
So H_{k+1} is true. By induction, H_n is true for all positive integers n. (States conclusion. Implication must be clearly expressed.)	**A1**
	(6)
$5\sum_{r=1}^{n} r^4 + \frac{1}{6}n(n+1)(2n+1) = \frac{1}{2}n^2(n+1)^2(2n+1)$	**M1**
$[5]\sum_{r=1}^{n} r^4 = \frac{1}{6}n(n+1)(2n+1)\left(3n(n+1) - 1\right)$	**M1**
$\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1)$ (CAO)	**A1**
	(3)
	Total: 9

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Q22

9231/11, 9231/13 • October/November 2021 • Question 3

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Mark scheme.

$a_2 = 2^3$ leading to $\ln a_2 = 3\ln 2$ so true when $n = 2$.	**B1**
Checks base case. Must be $n = 2$ and exact.	
Assume that $\ln a_k \geqslant 3^{k-1}\ln 2$.	**B1**
States inductive hypothesis.	
$\ln a_{k+1} = 3\ln\left(a_k + \frac{1}{a_k}\right) > 3\ln a_k$	**M1**
Applies result given in part (a).	
$\geqslant 3^k \ln 2$	**M1A1**
Applies inductive hypothesis and writes in required form.	
So true when $n = k+1$. By induction, true for all integers $n \geqslant 2$.	**A1**
States conclusion, all previous marks must be gained.	
	(6)
$3\ln\left(a_n + \frac{1}{a_n}\right) - \ln a_n > 2\ln a_n$	**M1**
Applies results given in parts (a) and (b).	
$> 2 \times 3^{n-1}\ln 2 = 3^{n-1}\ln 4$	**A1**
AG	
	(2)
	Total: 8

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Q23

9231/12 • October/November 2021 • Question 2

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Mark scheme.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = axe^{ax} + e^{ax} = (ax+1)e^{ax}$ so true when $n = 1$.	**M1A1**
Differentiates once using the product rule.	
Assume that $\dfrac{\mathrm{d}^k y}{\mathrm{d}x^k} = \left(a^k x + ka^{k-1}\right)e^{ax}$.	**B1**
States inductive hypothesis.	
$\dfrac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = a\left(a^k x + ka^{k-1}\right)e^{ax} + e^{ax}\left(a^k\right) = \left(a^{k+1} + (k+1)a^k\right)e^{ax}$	**M1A1**
Differentiates kth derivative.	
So true when $n = k+1$. By induction, true for all positive integers n.	**A1**
States conclusion.	

Total: 6

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Q24

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Mark scheme.

$u_1 > 4$ (given)	**B1**
Assume that $u_k > 4$ for some positive integer k.	**B1**
Then $u_{k+1} - 4 = \dfrac{u_k^2 + u_k + 12}{2u_k} - 4 = \dfrac{u_k^2 - 7u_k + 12}{2u_k}$	**M1**
$u_{k+1} - 4 = \dfrac{(u_k - 3)(u_k - 4)}{2u_k} > 0$	**A1**
Hence, by induction, $u_n > 4$ for all positive integers n.	**A1**
	(5)
$u_{n+1} - u_n = \dfrac{-u_n^2 + u_n + 12}{2u_n} = \dfrac{-(u_n - 4)(u_n + 3)}{2u_n}$	**M1 A1**
So $u_n > 4 \Rightarrow u_{n+1} - u_n < 0$	**A1**
	(3)
	Total: 8

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Q25

9231/11, 9231/13 • October/November 2022 • Question 2

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Mark scheme.

$7^2 + 97 - 50 = 96$ is divisible by 48	**B1**
Assume that $7^{2k} + 97^k - 50$ is divisible by 48 for some positive integer k.	**B1**
Then $7^{2k+2} + 97^{k+1} - 50 = (48+1)7^{2k} + (96+1)97^k - 50$	**M1A1**
is divisible by 48 because $48 \times 7^{2k} + 96 \times 97^k$ is divisible by 48.	**A1**
Hence, by induction, true for every positive integer n.	**A1**
	Total: 6

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Q26

9231/12 • October/November 2022 • Question 4

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Mark scheme.

$\dfrac{\mathrm{d}}{\mathrm{d}x}\big(x\mathrm{f}(x)\big) = x\mathrm{f}'(x) + \mathrm{f}(x) = x\mathrm{f}'(x) + (2(1) - 1)\mathrm{f}(x)$	**B1**
Assume true for $n = k$, so $\dfrac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}}\big(x\mathrm{f}(x)\big) = x\mathrm{f}'(x) + (2k-1)\mathrm{f}(x)$	**B1**
Then $\dfrac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}\big(x\mathrm{f}(x)\big) = x\mathrm{f}(x) + 2k\mathrm{f}'(x)$	**M1A1**
$\dfrac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}\big(x\mathrm{f}(x)\big) = x\mathrm{f}'(x) + \mathrm{f}(x) + 2k\mathrm{f}(x)$	**M1A1**
$= x\mathrm{f}'(x) + (2k+1)\mathrm{f}(x)$	
So, it is also true for $n = k + 1$. Hence, by induction, true for all positive integers.	**A1**
	Total: 7

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Q27

9231/11, 9231/12 • May/June 2023 • Question 1

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Mark scheme.

$2\mathbf{A} = \begin{pmatrix} 6 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2\times 3 & 0 \\ 3-1 & 2 \end{pmatrix}$ so true when $n = 1$.	**B1**
Assume that it is true for $n = k$, so $2\mathbf{A}^k = \begin{pmatrix} 2\times 3^k & 0 \\ 3^k - 1 & 2 \end{pmatrix}$.	**B1**
Then $2\mathbf{A}^{k+1} = \begin{pmatrix} 2\times 3^k & 0 \\ 3^k - 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2\times 3^{k+1} & 0 \\ 3^{k+1} - 3 + 2 & 2 \end{pmatrix}$	**M1A1**
So, it is also true for $n = k+1$. Hence, by induction, true for all positive integers.	**A1**
	(5)
$\det \mathbf{A}^n = \det \begin{pmatrix} 3^n & 0 \\ \frac{1}{2}(3^n - 1) & 1 \end{pmatrix} = 3^n$	**B1**
Or a multiple of $\mathbf{A}^{-n} = 2^{-1}3^{-n}\begin{pmatrix} 2 & 0 \\ 1-3^n & 2\times 3^n \end{pmatrix}$ seen.	
$\mathbf{A}^{-n} = 3^{-n}\begin{pmatrix} 1 & 0 \\ \frac{1}{2}(1-3^n) & 3^n \end{pmatrix}$	**B1**
	(2)
	Total: 7

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Q28

9231/13 • May/June 2023 • Question 1

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Mark scheme.

$5^3 + 32 - 33 = 124$ is divisible by 31.	**B1**
Assume that $5^{3k} + 32^k - 33$ is divisible by 31 for some positive integer k.	**B1**
$5^{3k+3} + 32^{k+1} - 33 = (124+1)5^{3k} + (31+1)32^k - 33$	**M1A1**
is divisible by 31 because $124 \times 5^{3k} + 31 \times 32^k$ is divisible by 31.	**A1**
Hence, by induction, true for every positive integer n.	**A1**
	Total: 6

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Q29

9231/11, 9231/13 • October/November 2023 • Question 2

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Mark scheme.

$1 = \dfrac{1-2x+x^2}{(1-x)^2} = \dfrac{(1-x)^2}{(1-x)^2}$ so H_1 is true.	**B1**
Assume that $\displaystyle\sum_{r=1}^{k} rx^{r-1} = \frac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2}$.	**B1**
$\displaystyle\sum_{r=1}^{k+1} rx^{r-1} = \frac{1-(k+1)x^k+kx^{k+1}}{(1-x)^2} + (k+1)x^k$	**M1**
$\dfrac{1-(k+1)x^k+kx^{k+1}+(k+1)x^k(1-2x+x^2)}{(1-x)^2}$	**M1**
$\dfrac{1+kx^{k+1}+(k+1)x^k(-2x+x^2)}{(1-x)^2} = \dfrac{1-(k+2)x^{k+1}+(k+1)x^{k+2}}{(1-x)^2}$	**A1**
So H_{k+1} is true. By induction, H_n is true for all positive integers n.	**A1**
	Total: 6

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Q30

9231/12 • October/November 2023 • Question 2

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Mark scheme.

$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2\mathrm{e}^x\right) = x^2\mathrm{e}^x + 2x\mathrm{e}^x = (x^2+2x)\mathrm{e}^x$ so true when $n = 1$.	**M1A1**
Assume that $\dfrac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^2\mathrm{e}^x\right) = \left(x^2 + 2kx + k(k-1)\right)\mathrm{e}^x$ [for some value of k].	**B1**
$\dfrac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(x^2\mathrm{e}^x\right) = \left(x^2 + 2kx + k(k-1)\right)\mathrm{e}^x + \mathrm{e}^x(2x+2k)$	**M1**
$\left(x^2 + 2(k+1)x + k(k+1)\right)\mathrm{e}^x$	**A1**
So true when $n = k+1$. By induction, true for all positive integers n.	**A1**
	Total: 6

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Q31

9231/11, 9231/12 • May/June 2024 • Question 2

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Mark scheme.

$6^4 + 38 - 2 = 1332$ is divisible by 74.	**B1**
Checks base case.	
Assume that $6^{4k} + 38^k - 2$ is divisible by 74 for some positive integer k.	**B1**
States inductive hypothesis.	
Then $6^{4k+4} + 38^{k+1} - 2 = (1295 + 1) \times 6^{4k} + (37 + 1) \times 38^k - 2$	**M1 A1**
Separates $6^{4k} + 38^k - 2$ or considers difference.	
Is divisible by 74 because $1295 \times 6^{4k} + 37 \times 38^k$ is divisible by 74.	**A1**
Hence, by induction, $6^{4k} + 38^k - 2$ is divisible by 74, is true for every positive integer n.	**A1**
	Total: 6

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Q32

9231/12 • October/November 2024 • Question 1

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Mark scheme.

$u_1 = 4 = 3^1 + 1$	**B1**
Assume that it is true for $n = k$, so $u_k = 3^k + 1$.	**B1**
Then $u_{k+1} = 3(3^k + 1) - 2 = 3^{k+1} + 1$	**M1A1**
[So, it is also true for $n = k + 1$]. Hence, by induction, $u_n = 3^n + 1$ for all positive integers.	**A1**
	Total: 5

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Q33

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Mark scheme.

$\dfrac{\mathrm{d}(\tan^{-1} x)}{\mathrm{d}x} = \dfrac{1}{1+x^2}$ so true when $n = 1$.	**B1**
Assume that $\dfrac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\tan^{-1} x\right) = P_k(x)\left(1+x^2\right)^{-k}$, where $\deg P_k(x) = k-1$.	**B1**
$\dfrac{\mathrm{d}^{k+1}(\tan^{-1} x)}{\mathrm{d}x^{k+1}} = P_k{}'(x)\left(1+x^2\right)^{-k} - 2kxP_k(x)\left(1+x^2\right)^{-k-1}$	**M1A1**
$= \left(P_k{}'(x)\left(1+x^2\right) - 2kxP_k(x)\right)\left(1+x^2\right)^{-k-1}$ so $\deg P_{k+1}(x) = k$	**A1**
So true when $n = k+1$. By induction, true for all positive integers n.	**A1**
	Total: 6

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Q34

9231/11, 9231/12 • May/June 2025 • Question 3

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Mark scheme.

$u_1 = 5 = 6^1 - 1$	**B1**
Shows base case.	
Assume that it is true for $n = k$, so $u_k = 6^k - 1$.	**B1**
States inductive hypothesis.	
Then $u_{k+1} = 6(6^k - 1) + 5 = 6^{k+1} - 1$.	**M1A1**
Substitutes into recursion formula.	
So, it is also true for $n = k + 1$. Hence, by induction, $u_n = 6^n - 1$ is true for all positive integers.	**A1**
States conclusion.	
	(5)
$\frac{u_{2n}}{u_n} = \frac{6^{2n} - 1}{6^n - 1} = \frac{(6^n - 1)(6^n + 1)}{6^n - 1} = 6^n + 1$	**M1A1**
Uses (a) and difference of two squares.	
	(2)
	Total: 7

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Q35

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Mark scheme.

$2025 + 47 - 2 = 2070 = 45 \times 46$ is divisible by 46.	**B1**
Checks base case.	
Assume that $2025^k + 47^k - 2$ is divisible by 46 for some positive integer k.	**B1**
States inductive hypothesis.	
Then $2025^{k+1} + 47^{k+1} - 2 = (2024+1)2025^k + (46+1)47^k - 2$	**M1A1**
Separates $2025^k + 47^k - 2$ or considers difference.	
is divisible by 46 because $2024 \times 2025^k + 46 \times 47^k = 46(44 \times 2025^k) + 46 \times 47^k$ is divisible by 46	**A1**
Shows that it is divisible by 46.	
Hence, [by induction], true for every positive integer n.	**A1**
Must reference the statement to be proven.	
	(6)
	Total: 6

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Q36

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Mark scheme.

$\dfrac{\mathrm{d}}{\mathrm{d}x}(x\cos x) = \cos x - x\sin x = (-1)^1\,(x\sin x - (2(1)-1)\cos x)$	**B1**
Checks base case using product rule.	
Assume true for $n = k$, so $\dfrac{\mathrm{d}^{2k-1}}{\mathrm{d}x^{2k-1}}(x\cos x) = (-1)^k\,(x\sin x - (2k-1)\cos x)$	**B1**
States inductive hypothesis.	
Then $\dfrac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}(x\cos x) = (-1)^k\,(x\cos x + 2k\sin x)$	**M1A1**
Differentiates once.	
$\dfrac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}(x\cos x) = (-1)^k\,(-x\sin x + \cos x + 2k\cos x) = (-1)^{k+1}\,(x\sin x - (2k+1)\cos x)$	**M1A1**
Differentiates again.	
So, it is also true for $n = k+1$. Hence, [by induction], $\dfrac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\cos x) = (-1)^n\,(x\sin x - (2n-1)\cos x)$ is true for all positive integers.	**A1**
States conclusion.	
	Total: 7

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Q37

9231/12 • October/November 2025 • Question 3

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Mark scheme.

$u_1 < 5$ (given)	**B1**
States base case.	
Assume that $u_k < 5$ for some positive integer k.	**B1**
States inductive hypothesis.	
Then $5 - u_{k+1} = 5 - \dfrac{6u_k + 5}{u_k + 2} = \dfrac{5u_k + 10 - 6u_k - 5}{u_k + 2}$	**M1**
Considers $5 - u_{k+1}$, puts over common denominator.	
$= \dfrac{5 - u_k}{u_k + 2} > 0.$	**A1**
Hence, by induction, $u_n < 5$ for all positive integers n.	**A1**
	(5)
$u_{n+1} - u_n = \dfrac{6u_n + 5}{u_n + 2} - u_n = \dfrac{6u_n + 5 - u_n^2 - 2u_n}{u_n + 2} = \dfrac{-u_n^2 + 4u_n + 5}{u_n + 2}$	**M1A1**
Considers $u_{n+1} - u_n$ or uses $5 - u_{n+1} = \dfrac{5 - u_n}{u_n + 2}$.	
$= \dfrac{(5 - u_n)(1 + u_n)}{u_n + 2}$. So $u_n < 5 \Rightarrow u_{n+1} - u_n > 0$	**A1**
Uses $u_n < 5$.	
	(3)
	Total: 8

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Q38

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Mark scheme.

LHS $= \left(\frac{6}{5}\right)^1 = \frac{6}{5}$, RHS $= 1 + \frac{1}{5} = \frac{6}{5}$	**B1**
Checks base case.	
Assume that $\left(\frac{6}{5}\right)^k \geqslant 1 + \frac{1}{5}k$ for some positive integer k.	**B1**
States inductive hypothesis.	
Then $\left(\frac{6}{5}\right)^{k+1} \geqslant \left(\frac{6}{5}\right)\left(1 + \frac{1}{5}k\right)$	**M1**
Moving to the next term and using the inductive hypothesis.	
$= \frac{6}{5} + \frac{6}{25}k \geqslant \frac{6}{5} + \frac{1}{5}k = 1 + \frac{1}{5}(k+1)$	**A1**
Convincing algebra to reach the required form.	
Hence, by induction, $\left(\frac{6}{5}\right)^n \geqslant 1 + \frac{1}{5}n$ is true for every positive integer n.	**A1**
Must follow M1 A1.	
	Total: 5

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